Volume Computing

1. Cross section technique. A solid $T$ is placed along an axis ($x$-axis or $y$-axis, assume it to be $x$-axis) from lower bound $a$ and upper bound $b$. If for each value $x$ between $a$ and $b$, the area of the cross section of $T$ is $A(x)$, then

$$\text{Volume of } T = \int_a^b A(x) \, dx.$$ 

Note that $A(x) \, dx$ represents the volume of a small slice of $T$ while $\int_a^b A(x) \, dx$ means adding up all such small pieces in the Riemann sum sense under a limiting process.

Determine the bounds of integration. These bounds are the coordinates of the ends of the solid. Suppose a solid $T$ is obtained from rotating a region $R$ about an axis. If the axis of rotation is parallel to the $x$-axis, then the integration bounds are $x$-bounds; If the axis of rotation is parallel to the $y$-axis, then the integration bounds are $y$-bounds.

2. Cylindrical shell technique. A cylindrical shell with radius $r$, height $h$ and thickness $\Delta$ has volume $2\pi dh \Delta$. To apply the shell technique to compute a volume, we first partition the solid into small shells and then add up all the volumes of the shells in the Riemann sum sense under a limiting process. Therefore, a generic form of the shell technique is

$$\int_a^b 2\pi \text{(distance from the shell to the axis of rotation)} \times \text{(height of the shell)} \times \text{(thickness)} \, dx.$$ 

Determine the bounds of integration. This is different from the cross section technique. Suppose a solid $T$ is obtained from rotating a region $R$ about an axis. If the axis of rotation is parallel to the $x$-axis, then the integration bounds are $y$-bounds; If the axis of rotation is parallel to the $y$-axis, then the integration bounds are $x$-bounds.

Example 1 Find volume of the solid obtained by rotating the region $R$ bounded by $y = 9 - x^2$ and $y = 0$ about $x$-axis.

Solution: We use cross section technique. First determine the integration bounds. As the axis of rotation is the $x$-axis, the bounds should be the $x$-coordinates of the ends of $R$. Note that the curves $y = 9 - x^2$ and $y = 0$ intersect at $x = -3$ and $x = 3$, and so lower bound $a = -3$ and upper bound $b = 3$.

For each $x$ with $-3 \leq x \leq 3$, the cross section is a circle with radius $9 - x^2$, (the $y$ value of the curve bounded above region $R$ at $x$), and so $A(x) = \pi(9 - x^2)^2$. Thus

$$\text{Volume} = \pi \int_{-3}^3 (9 - x^2)^2 \, dx = \pi \int_{-3}^3 (81 - 18x^2 + x^4) \, dx = \frac{1296}{5} \pi.$$ 

1
**Example 2** Find volume of the solid obtained by rotating the region $R$ bounded by $y = 1 - x^2$ and $y = 0$ about the line $x = 2$.

**Cross Section Solution:** As the axis of rotation is parallel to the $y$-axis, the bounds should be the $y$-coordinated of the ends of $R$. Note that the curves $y = 1 - x^2$ and $y = 0$ bound the region from above and from below, respectively, and so lower bound $c = 0$ and upper bound $d = 1$.

For each $y$ with $0 \leq y \leq 1$, the cross section of the solid at $y$ is an annular ring with the bigger radius $r_2 = 2 + \sqrt{1 - y}$ and smaller radius $r_1 = 2 - \sqrt{1 - y}$. Thus the cross section area at $y$ is

$$A(y) = \pi r_2^2 - \pi r_1^2 = \pi (r_2^2 - r_1^2) = \pi \left[ (2 + \sqrt{1 - y})^2 - (2 - \sqrt{1 - y})^2 \right] = 8\pi \sqrt{1 - y}.$$  

It follows that the volume of the solid is

$$\text{Volume} = 8\pi \int_0^1 \sqrt{1 - y} dy = 8\pi \left[ \frac{2(1 - y)^{3/2}}{3} \right]_0^1 = \frac{16}{3}\pi.$$ 

**Shell Technique Solution:** As the axis of rotation is parallel to the $y$-axis, the bounds should be the $x$-coordinated of the ends of $R$ for the shell technique. Note that the curves $y = 1 - x^2$ and $y = 0$ intersect at $x = -1$ and $x = 1$, and so lower bound $a = -1$ and upper bound $b = 1$.

For each $x$ with $-1 \leq x \leq 1$, the shell generated at $x$ has radius $2 - x$, height $1 - x^2$, and thickness $dx$, and so the volume of this shell at $x$ is

$$2\pi(2 - x)(1 - x^2)dx = 2\pi(2 - x - 2x^2 + x^3)dx.$$  

It follows that the volume of the solid is (using properties of even and odd functions integrating on a symmetric interval)

$$\text{Volume} = 2\pi \int_{-1}^1 (2 - x - 2x^2 + x^3)dx = 4\pi \left[ \frac{2x - 2x^3}{3} \right]_0^1 = \frac{16}{3}\pi.$$