Compute the higher derivatives

**Facts** For a function $f$, the derivative of $f'$, denoted $f''$, is the second derivative of $f$; and the derivative of $f''$, denoted $f'''$ or $f^{(3)}$, is the third derivative of $f$. With the notation

$$ f'(x) = D_x(f(x)) = \frac{df}{dx}, \quad f''(x) = D_x(f'(x)) = D^2_x(f(x)) = \frac{d^2f}{dx^2}, \quad f^{(3)}(x) = D_x(f''(x)) = D^3_x(f(x)) = \frac{d^3f}{dx^3}. $$

we define the $n$th derivative of $f(x)$ to be

$$ f^{(n)}(x) = D^n_x(f(x)) = \frac{d^n f}{dx^n}. $$

**Example 1** Compute the first three derivatives of $f(x) = 2x^4 - 3x^3 + 6x - 17$.

**Solution**: Compute the derivatives term by term to get

$$ f'(x) = 8x^3 - 9x^2 + 6 $$
$$ f''(x) = 24x^2 - 18x $$
$$ f'''(x) = 48x - 18. $$

**Example 2** Compute the first three derivatives of $f(x) = 2x^5 + x^{\frac{3}{2}} - \frac{1}{2x}$.

**Solution**: Compute the derivatives term by term to get

$$ f'(x) = 10x^4 + \frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2x^2} $$
$$ f''(x) = 40x^3 + \frac{3}{4}x^{-\frac{1}{2}} - \frac{1}{x^3} $$
$$ f'''(x) = 120x^2 - \frac{3}{8}x^{-\frac{3}{2}} + \frac{3}{x^4}. $$

**Example 3** Compute the first three derivatives of $f(x) = \sin(x) \cos(x)$.

**Solution**: Apply product rule to get

$$ f'(x) = \cos^2(x) - \sin^2(x) $$
$$ f''(x) = -2\cos(x)\sin(x) - 2\cos(x)\sin(x) = -4\cos(x)\sin(x) $$
$$ f'''(x) = -4(\cos^2(x) - \sin^2(x)). $$

**Example 4** Compute the first three derivatives of $f(x) = x^2 \cos(x)$. 
Solution: Apply product rule to get

\[ f'(x) = 2x \cos(x) - x^2 \sin(x) \]
\[ f''(x) = 2 \cos(x) - 2x \sin(x) - 2x \sin(x) + 2x^2 \cos(x) = 2 \cos(x) - 4x \sin(x) + x^2 \cos(x) \]
\[ f'''(x) = -2 \sin(x) - 4 \sin(x) - 4x \cos(x) + 2x \cos(x) - x^2 \sin(x) \]
\[ = -6 \sin(x) - 2x \cos(x) - x^2 \sin(x) \]

Example 5 Compute the first three derivatives of \( f(x) = x \sqrt{x+1} \).

Solution: Write \( f(x) = (x + 1)^{\frac{1}{2}} \). Apply product rule and chain rule in each step below.

\[ f'(x) = (x + 1)^{\frac{1}{2}} + \frac{1}{2} x (x + 1)^{-\frac{1}{2}} \]
\[ f''(x) = \frac{1}{2} (x + 1)^{-\frac{1}{2}} + \frac{1}{2} (x + 1)^{-\frac{3}{2}} - \frac{1}{4} x (x + 1)^{-\frac{3}{2}} = (x + 1)^{-\frac{1}{2}} - \frac{1}{4} x (x + 1)^{-\frac{3}{2}} \]
\[ f'''(x) = -\frac{1}{2} (x + 1)^{-\frac{3}{2}} - \frac{1}{4} (x + 1)^{-\frac{3}{2}} + \frac{3}{8} x (x + 1)^{-\frac{3}{2}} = -\frac{3}{4} (x + 1)^{-\frac{3}{2}} + \frac{3}{8} x (x + 1)^{-\frac{3}{2}}. \]

Example 6 Given \( \sin(y) = xy \), compute \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \).

Solution: We need to use implicit differentiation. View \( y = y(x) \) and differentiate both sides of the equation \( \sin(y) = xy \) with respect to \( x \).

\[ \cos(y) y' = y + xy', \text{ and so } \frac{dy}{dx} = y' = \frac{y}{\cos(y) - x}. \]

To compute \( \frac{d^2y}{dx^2} \) is to differentiate both sides of the equation \( \cos(y) y' = y + xy' \) with respect to \( x \). This yields

\[ -\sin(y) y' + \cos(y) y'' = y' + y + xy''. \]

Substitute \( y' = \frac{y}{\cos(y) - x} \) to get

\[ \frac{-y \sin(y)}{\cos(y) - x} + \cos(y) y'' = \frac{2y}{\cos(y) - x} + xy'', \]

and so \( (\cos(y) - x)y'' = \frac{2y}{\cos(y) - x} + \frac{y \sin(y)}{\cos(y) - x} \). Hence

\[ y'' = \frac{2y}{\cos(y) - x} + \frac{y \sin(y)}{\cos(y) - x} = \frac{2y + y \sin(y)}{(\cos(y) - x)^2}. \]
Example 7 Given $x^2 + xy + y^2 = 3$, compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

**Solution:** We need to use implicit differentiation. View $y = y(x)$ and differentiate both sides of the equation $x^2 + xy + y^2 = 3$ with respect to $x$.

$$2x + y + xy' + 2yy' = 0,$$ and so

$$\frac{dy}{dx} = y' = -\frac{2x - y}{x + 2y}.$$

One way to compute $\frac{d^2y}{dx^2}$ is to differentiate both sides of the equation $2x + y + xy' + 2yy' = 0$ with respect to $x$. This yields

$$2 + y' + xy'' + 2(y')^2 + 2yy'' = 0,$$ or

$$2 + 2y' + 2(y')^2 + xy'' + 2yy'' = 0.$$

Substitute $y' = -\frac{2x - y}{x + 2y}$ to get

$$2 + 2\left(-\frac{2x - y}{x + 2y}\right) + 2\left(-\frac{2x - y}{x + 2y}\right)^2 + xy'' + 2yy'' = 0.$$

It follows that

$$(x + 2y)y'' = -2\frac{(x + 2y)^2 - (2x + y)(x + 2y) + 1}{(x + 2y)^2}.$$

Thus

$$y'' = -2\frac{(x + 2y)^2 - (2x + y)(x + 2y) + 1}{(x + 2y)^3}.$$

Another way to compute $\frac{d^2y}{dx^2}$ is to differentiate $y'$ (using quotient rule below):

$$y'' = D_x(y') = \frac{(-2 + y')(x + 2y) - (1 - 2y')(-2x - y)}{(x + 2y)^2} = \frac{(-2 + \frac{-2x - y}{x + 2y})(x + 2y) + (1 - 2\frac{-2x - y}{x + 2y})(2x + y)}{(x + 2y)^2} = \frac{-2(x + 2y)^2 - (2x + y)(x + 2y) + 1}{(x + 2y)^3}.$$

Example 8 Compute the first three derivatives of $f(x) = \frac{\sin(x)}{x}$.

**Solution:** Apply quotient rule in computing $f'$, and quotient and product rules in computing $f''$ and $f'''$.

$$f'(x) = \frac{x \cos(x) - \sin(x)}{x^2}$$

$$f''(x) = \frac{(\cos(x) - x \sin(x) - \cos(x))x^2 - 2x(x \cos(x) - \sin(x))}{x^4} = -\frac{x^2 \sin(x) - 2x \cos(x) + 2 \sin(x)}{x^3}$$

$$f'''(x) = \frac{(-2x \sin(x) - x^2 \cos(x) - 2 \cos(x) + 2x \sin(x) + 2 \cos(x))x^3}{x^6} - \frac{3x^2(-x^2 \sin(x) - 2x \cos(x) + 2 \sin(x))}{x^6}$$

$$= \frac{-x^3 \cos(x) + 3x^2 \sin(x) + 6x \cos(x) - 6 \sin(x)}{x^4}.$$