Determine monotone intervals of a function

Facts: Let \( f(x) \) be a function on an interval \( I \).

(1) If for any pair of points \( x_1, x_2 \) in \( I \) with \( x_1 < x_2 \) we always have \( f(x_1) > f(x_2) \) (respectively, \( f(x_1) < f(x_2) \)) then \( f(x) \) is decreasing (respectively, increasing) in the interval \( I \).

(2) If \( f'(x) > 0 \) (respectively, \( f'(x) < 0 \)) for all \( x \) in \( I \), then \( f(x) \) is decreasing (respectively, increasing) in the interval \( I \).

(3) To determine the monotone intervals of \( f \) (the intervals in which \( f(x) \) is either always increasing or always decreasing), we can use the following process.

   (Step 1) Compute \( f'(x) \), and find the points at which \( f'(x) = 0 \) or \( f'(x) \) does not exist. Let \( c_1, c_2, \ldots \) denote these points.

   (Step 2) These points \( c_1, c_2, \ldots \) will partition the domain of \( f(x) \) into intervals. Determine the sign of \( f'(x) \) in each of the intervals and then apply (2) to make conclusions.

Example 1 Determine the open intervals in which the function \( f(x) = 2x - \frac{1}{6}x^2 - \frac{1}{9}x^3 \) is increasing and those in which \( f(x) \) is decreasing.

Solution: Note that the domain of \( f(x) \) is \((-\infty, \infty)\).

(Step 1) Compute \( f'(x) = 2 - \frac{1}{3}x - \frac{1}{3}x^2 \). Set \( f'(x) = 2 - \frac{1}{3}x - \frac{1}{3}x^2 = 0 \). Use 3 as a common denominator to get

\[
\frac{6 - x - x^2}{3} = 0, \quad \text{and so} \quad (2 - x)(3 + x) = 0.
\]

Thus \( c_1 = -3 \) and \( c_2 = 2 \) are the critical points.

(Step 2) The two points \(-3\) and \(2\) partitioned the domain of \( f(x) \) into intervals \((-\infty, -3)\), \((-3, 2)\) and \((2, \infty)\).

Since \( f'(-4) < 0 \), \( f'(0) > 0 \) and \( f'(3) < 0 \), we conclude that \( f'(x) < 0 \) in both \((-\infty, -3)\) and \((2, \infty)\), and that \( f'(x) > 0 \) in \((-3, 2)\). Therefore, \( f(x) \) is decreasing in both \((-\infty, -3)\) and \((2, \infty)\), and \( f(x) \) is increasing in \((-3, 2)\).

Example 2 Determine the open intervals in which the function \( f(x) = \frac{x}{x+1} \) is increasing and those in which \( f(x) \) is decreasing.

Solution: Note that the domain of \( f(x) \) is \((-\infty, -1)\) and \((-1, \infty)\).

(Step 1) Compute \( f'(x) = \frac{-1}{(x+1)^2} \). Thus \( f'(x) > 0 \) for any \( x \) in the domain of \( f(x) \).

(Step 2) Therefore, \( f(x) \) is decreasing in both \((-\infty, -1)\) and \((-1, \infty)\).
Example 3  Determine the open intervals in which the function $f(x) = \frac{(x-1)^2}{x^2-3}$ is increasing and those in which $f(x)$ is decreasing.

Solution: Note that the domain of $f(x)$ is $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, \sqrt{3})$ and $(\sqrt{3}, \infty)$.
(Step 1) Compute

$$f'(x) = \frac{2(x-1)(x^2-3) - 2x(x-1)^2}{(x^2-3)^2} = \frac{(2x^3 - 2x^2 - 6x + 6) - (2x^3 - 4x^2 + 2x)}{(x^2 - 3)^2} = \frac{2(x^2 - 4x + 3)}{(x^2 - 3)^2} = \frac{2(x-1)(x-3)}{(x^2 - 3)^2}.$$

Thus $c_1 = 1$ and $c_2 = 3$ are the critical points.
(Step 2) The two points 1 and 3 partitioned the domain of $f(x)$ into intervals $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, 1)$, $(1, \sqrt{3})$, $(\sqrt{3}, 3)$ and $(3, \infty)$.

Since $f'(-4) > 0$, $f'(0) > 0$, $f'(1.5) < 0$, $f'(2) < 0$, and $f'(4) > 0$, we conclude that $f'(x) < 0$ in the intervals $(1, \sqrt{3})$ and $(\sqrt{3}, 3)$, and that $f'(x) > 0$ in $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, 1)$, and $(3, \infty)$. Therefore, $f(x)$ is decreasing in both $(1, \sqrt{3})$ and $(\sqrt{3}, 3)$, and $f(x)$ is increasing in $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, 1)$, and $(3, \infty)$.  

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