Compute the differential and linear approximation of a function

**Facts:** Let \( f(x) \) be a differentiable function, and \( x_0 \) be a point in the domain of \( f(x) \).

1. The **differential** of \( f(x) \) is
   \[
   df = f'(x)\,dx.
   \]
   When \( y = f(x) \), we also use \( dy \) for \( df \).

2. A change in \( x \), called the \( x \)-increment, is \( \Delta x \), which is also denoted \( dx \). The corresponding changes in \( y = f(x) \), is
   \[
   \Delta y = f(x + \Delta x) - f(x).
   \]

3. Using the differential \( dy \) to approximate \( \Delta y \) is call the **linear approximation** of the function \( f \) (near the point \( x \)).

4. Use linear approximation to estimate the value of the function \( f(x) \) near \( x = x_0 \) with given value of \( \Delta x \):
   \[
   f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x = f(x_0) + df(x_0).
   \]
   The linear approximation of \( f(x) \) near \( x_0 \) is often written as \( L(x) = f(x_0) + f'(x_0)\Delta x \).

5. Estimate the change of the function by the linear approximation of function \( f(x) \) near \( x = x_0 \) with given value of \( \Delta x \):
   \[
   \text{error} = \Delta f(x_0) = f(x_0 + \Delta x) - f(x_0) = f'(x_0)\Delta x = df(x_0).
   \]

**Example 1** Compute the differential of \( y = \cos \sqrt{x} \).

**Solution:** First compute \( y' = -\sin \sqrt{x} \cdot \frac{1}{2\sqrt{x}} \). Thus the answer is
\[
 dy = -\sin \sqrt{x} \cdot \frac{1}{2\sqrt{x}} \, dx = -\frac{\sin \sqrt{x}}{2\sqrt{x}} \, dx.
\]

**Example 2** Given \( f(x) = (1 - 2x)^\frac{3}{2} \), find \( L(x) \) near \( x = 0 \).

**Solution:** This is the case when \( x_0 = 0 \) in Fact (4) above. Compute \( f'(x) = (-2)^\frac{3}{2}(1 - 2x)^\frac{3}{2} = -3(1 - 2x)^\frac{3}{2} \). Note that \( f(0) = -3 \). Therefore the answer is
\[
 L(x) = f(0) + f'(0)\,dx = 1 - 3\,dx.
\]

**Example 3** Use a linear approximation to estimate the number \( \sqrt{80} \).

**Solution:** Let \( f(x) = \sqrt{x} \). Then \( x_0 + \Delta x = 80 \) in Fact (4) above. Notice that \( \sqrt{81} = 9 \), we let
$x_0 = 81$ and then $\Delta x = 80 - 81 = -1$. Note that $f'(x) = \frac{1}{2\sqrt{x}}$. Thus $f'(x_0) = f'(81) = \frac{1}{2\sqrt{81}} = \frac{1}{18}$. Apply (4) to get the answer

$$\sqrt{80} \simeq \sqrt{81} + \frac{1}{18}(-1) = 9 - \frac{1}{18} = \frac{161}{18}.$$  

**Example 4**  Use a linear approximation to estimate the number $\sin 32^\circ$.

**Solution:** Let $f(x) = \sin(x)$. Then $x_0 + \Delta x = 32^\circ$. Note that $\sin 30^\circ = \sin \frac{\pi}{6} = \frac{1}{2}$, and so we let $x_0 = 30^\circ = \frac{\pi}{6}$. Thus $\Delta x = 32^\circ - 30^\circ = 2^\circ$. When using calculus to deal with trig function values, it is recommended to convert the measure of an angle from degrees to radians. Thus

$$2^\circ = \frac{2^\circ \cdot \pi}{180^\circ} = \frac{\pi}{90}.$$  

Note that $f'(x) = \cos(x)$ and that $f'(x_0) = f'(\frac{\pi}{6}) = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$. Apply (4) to get the answer

$$\sin 32^\circ \simeq \sin 30^\circ + \frac{\sqrt{3}}{2} \cdot \frac{\pi}{90} = \frac{1}{2} + \frac{\pi \sqrt{3}}{180} = \frac{90 + \pi \sqrt{3}}{180}.$$  

**Example 5**  Use a linear approximation to estimate the change of the area of a square, when its edge length is decreased from 10 in to 9.8 in.

**Solution:** Let $x$ denote the length of an edge of the square. Then the area is $f(x) = x^2$. Note that $x$ changes from 10 in to 9.8 in, and so we set $x_0 = 10$ and $x_0 + \Delta x = 9.8$. It follows that $\Delta x = 9.8 - 10 = -0.2$. Note that $f'(x) = 2x$. Thus $f'(x_0) = f'(10) = 20$. Apply (5) to get the estimated change as

$$\Delta f = f'(10)\Delta x = (20)(-0.2) = -4.$$  

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