Find maximum and minimum values of a function over a closed interval

**Facts:** Let \( f(x) \) be a function on \([a, b]\) and \(c\) is a point in the interval \([a, b]\).

1. If for any point \(x\) in \([a, b]\), \(f(x) \geq f(c)\) (respectively, \(f(x) \leq f(c)\)), then \(f(c)\) is the **absolute (or global) minimum value** (respectively, **absolute (or global) local maximum value**) of \(f(x)\) on \([a, b]\).

2. If \(a < c < b\), and for any point \(x\) in an open interval containing \(c\), \(f(x) \geq f(c)\) (respectively, \(f(x) \leq f(c)\)), then \(f(c)\) is a **local minimum value** of \(f(x)\) (respectively, **local maximum value**) on \([a, b]\).

3. If \(f(x)\) is continuous on \([a, b]\) and differentiable in \((a, b)\), a point \(c\) in \([a, b]\) is a **critical point** of \(f(x)\) if either \(f'(c)\) does not exist, or \(f'(c) = 0\).

4. **Important:** If \(f(x)\) is continuous on \([a, b]\) and differentiable in \((a, b)\), and if for some \(c\) in \((a, b)\), \(f(c)\) is a local maximum or local minimum, then \(c\) must be a critical point. Any absolute maximum or minimum must take place at critical points inside the interval or at the boundaries point \(a\) or \(b\).

**Example 1** State whether the function \(f(x) = |x - 2|\) attains a maximum value or a minimum value in the interval \((1, 4]\).

**Solution:** Apply the definition of absolute value to get

\[
f(x) = \begin{cases} 
x - 2 & \text{if } 2 \leq x \leq 4, \\
2 - x & \text{if } 1 < x < 2.
\end{cases}
\]

Thus the graph of this function consists of two pieces of lines, and so the minimum value \(f(2) = 0\) @ \(x = 2\), and the maximum value is \(f(4) = 2\) @ \(x = 4\).

**Example 2** Find the maximum value and the minimum value attained by \(f(x) = \frac{1}{x(1-x)}\) in the interval \([2, 3]\).

**Solution:** Note that the domain of \(f(x)\) does not contain \(x = 0\) and \(x = 1\), and these points are not in the interval \([2, 3]\).

(Step 1) Find critical points. Compute

\[
f'(x) = -\frac{1 - 2x}{x^2(1-x)^2} = \frac{2x - 1}{x^2(1-x)^2}.
\]
Therefore, the only possible critical point is \( x = \frac{1}{2} \). As this point is not in the interval \([2, 3]\), it is not a critical point.

(Step 2) Compute \( f(x) \) at the critical point(s) and at the boundaries of the closed interval. 

\[
\begin{align*}
f(2) &= \frac{1}{2(1-2)} = -\frac{1}{2}, \\
f(3) &= \frac{1}{3(1-3)} = -\frac{1}{6}.
\end{align*}
\]

(Step 3) Compare the data resulted in Step 2 to make conclusions.

\( f(x) \) attains its absolute maximum value \( f(3) = -\frac{1}{6} \) @ \( x = 3 \) and \( f(x) \) attains its absolute minimum value \( f(2) = -\frac{1}{2} \) @ \( x = 2 \).

**Example 3**

Find the maximum value and the minimum value attained by \( f(x) = x^2 + \frac{1}{6}x \) in the interval \([1, 3]\).

**Solution:** Note that the domain of \( f(x) \) does not contain \( x = 0 \), and this point is not in the interval \([1, 3]\).

(Step 1) Find critical points. Compute

\[
f'(x) = 2x - \frac{16}{x^2} = \frac{2x^3 - 16}{x^2}.
\]

Set \( f'(x) = 0 \). As a fraction equals zero if and only if its numerator equals zero, we have \( 2x^3 - 16 = 0 \), and so the only possible critical point is \( x = 2 \). As this point is in the interval \([1, 3]\), it is a critical point.

(Step 2) Compute \( f(x) \) at the critical point(s) and at the boundaries of the closed interval.

\[
\begin{align*}
f(1) &= 1 + \frac{16}{1} = 17, \\
f(2) &= 2^2 + \frac{16}{2^2} = 8, \\
f(3) &= 3^2 + \frac{16}{3^2} = 9 + \frac{16}{9} = \frac{97}{9}.
\end{align*}
\]

(Step 3) Compare the data resulted in Step 2 to make conclusions.

Note that \( 8 < \frac{97}{9} < 17 \), and so \( f(x) \) attains its absolute maximum value \( f(1) = 17 \) @ \( x = 1 \) and \( f(x) \) attains its absolute minimum value \( f(2) = 8 \) @ \( x = 2 \).
**Example 4** Find the maximum value and the minimum value attained by $f(x) = x^{\frac{1}{2}} - x^{\frac{3}{2}}$ in the interval $[0, 4]$.

**Solution**: Note that the domain of $f(x)$ does not contain any negative number, and so the function is continuous on $[0, 4]$.

(Step 1) Find critical points. Compute

$$f'(x) = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{2\sqrt{x}} - \frac{3\sqrt{x}}{2} = \frac{1 - 3x}{2\sqrt{x}}.$$

Set $f'(x) = 0$. As a fraction equals zero if and only if its numerator equals zero, we have $1 - 3x = 0$, and so the $x = \frac{1}{3}$ is a critical point. Since $f'(x)$ does not exist at $x = 0$, but $f(x)$ is (right) continuous at $x = 0$, both $\frac{1}{3}$ and 0 are critical points.

(Step 2) Compute $f(x)$ at the critical point(s) and at the boundaries of the closed interval.

$$f(0) = 0 - 0 = 0,$$
$$f\left(\frac{1}{3}\right) = \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{1}{2\sqrt{3}},$$
$$f(4) = 4^{\frac{1}{2}} - 4^{\frac{3}{2}} = 2 - 8 = -6.$$

(Step 3) Compare the data resulted in Step 2 to make conclusions.

Note that $-6 < 0 < \frac{1}{2\sqrt{3}}$, and so $f(x)$ attains its absolute maximum value $f\left(\frac{1}{3}\right) = \frac{1}{2\sqrt{3}}$ @ $x = \frac{1}{3}$ and $f(x)$ attains its absolute minimum value $f(4) = -6$ @ $x = 4$. 