7. Nilpotent and Solvable Groups

(7.1) **(The Central Series)** Let $G$ be a group, and let $C(G)$ denote the center of $G$; and define

\[
C_0(G) = \{e\}
C_1(G) = C(G)
C_2(G) = \text{inverse image of } C(G/C_1(G)) \text{ under the projection } G \mapsto G/C_1(G)
\cdots = \cdots
C_i(G) = \text{inverse image of } C(G/C_{i-1}(G)) \text{ under the projection } G \mapsto G/C_{i-1}(G)
\cdots = \cdots
\]

Then, each $C_i(G) \leq G$, and the series

\[
\{e\} < C_1(G) < C_2(G) \cdots
\]

is called the **ascending central series** of $G$. A group $G$ is **nilpotent** if for some integer $n$, $G_n = G$.

(7.1a) If $G_n = G$, then $G_{n+1} = G$; and for each $i$, $C(G/C_i(G)) = C_{i+1}(G)/C_i(G)$.
(7.1b) Any abelian group is nilpotent.
(7.1c) (Thm 7.2) Every finite $p$-group is nilpotent. (By the class equation).
(7.1d) (Thm 7.3) The direct product of a finite number of nilpotent groups is nilpotent.

**Proof:** It suffices to show the case when the factor number is 2.

(i) If $H_1 \leq H$ and $K_1 \leq K$, then $H/H_1 \times K/K_1 \cong (H \times K)/(H_1 \times K_1)$.
(ii) $C_i(H) \times C_i(K) = C_i(H \times K)$

As (i) is a special case of (1.8) in Products of Groups, we only need to prove (ii). Note first that

\[
C_i(H \times K) = C(H \times K) = C(H) \times C(K) = C_1(H) \times C_1(K).
\]

In particular, for any $i$,

\[
C(H/C_i(H) \times K/C_i(K)) = C(H/C_i(H)) \times C(K/C_i(K)).
\]
Fixed an \( i \geq 1 \). Assume inductively that
\[
C_i(H \times K) = C_i(H) \times C_i(K).
\]

Then \( C_i(H) \triangleleft H \) and \( C_i(K) \triangleleft K \). Let \( \pi_H : H \mapsto H/C_i(H) \) and \( \pi_K : K \mapsto K/C_i(K) \) be the canonical projections, and \( \pi = (\pi_H, \pi_K) : H/C_i(H) \times K/C_i(K) \) be the homomorphism defined componentwise. By (i), there is an isomorphism
\[
\psi : H/C_i(H) \times K/C_i(K) \mapsto (H \times K)/(C_i(H) \times C_i(K)) = (H \times K)/C_i(H \times K).
\]

Let \( \phi = \psi \cdot \pi : H \times K \mapsto (H \times K)/C_i(H \times K) \). Then
\[
C_{i+1}(H \times K) = \phi^{-1}[C((H \times K)/C_i(H \times K))] = \pi^{-1}\psi^{-1}[C((H \times K)/C_i(H \times K))]
= \pi^{-1}[C(H/C_i(H)) \times K/C_i(K)]
= \pi^{-1}[C(H/C_i(H))] \times \pi_K^{-1}[C(K/C_i(K))]
= C_{i+1}(H) \times C_{i+1}(K).
\]

Thus by induction, (ii) always holds.

Suppose that both \( H \) and \( K \) are nilpotent. Then
\[
\{e_H\} < C_1(H) < C_2(H) < \cdots < C_m(H) = H,
\]
and
\[
\{e_K\} < C_1(K) < C_2(K) < \cdots < C_n(K) = K.
\]
We may assume that \( m \geq n \). Then \( C_{n+1}(K) = \cdots C_m(K) = K \) and so,
\[
\{e_H\} \times \{e_K\} < C_1(H) \times C_1(K) < \cdots < C_m(H) \times C_m(K).
\]
By (ii), (only the last equality is needed),
\[
\{e_{H \times K}\} < C_1(H \times K) < \cdots < C_m(H \times K) = H \times K.
\]

(7.2) (Lemma 7.4) Let \( G \) be nilpotent, and let \( H < G \) be proper. Then \( H < N_G(H) \) is also proper.

**Proof**: \( G \) is nilpotent \( \implies G_m = G \). Let \( n \) be the largest such that \( G_n < H \). Since
$H \neq G, n < m$. Pick $a \in C_{n+1}(H) - H$. Then $a \in N_G(H) - H$.

(4.3) If $|G| < \infty$, and if $P \in Syl_p(P)$, then $N_G(N_G(P)) = N_G(P)$.

**Proof:** This is a special case of (3.1). A direct proof is as follows. $P \leq N_G(P)$. Since $\forall x \in N_G(N_G(P)) \implies x(N_G(P))x^{-1} = N_G(P)$, $xPx^{-1} = P \implies x \in N_G(P)$.

(7.4) (Prop. 7.5) A finite group is nilpotent if and only if $G$ is the direct product of its Sylow subgroups.

**Proof:** (Sufficiency) If $G$ is the direct product of its Sylow subgroups, then by (7.1c) and (7.1d), $G$ is nilpotent.

(Necessity) Let $G$ be nilpotent. If $G$ itself is a $p$-group, then done. Assume that $G$ is not a $p$-group.

**Step 1** Every Sylow $p$-subgroup is normal in $G$.

Let $P \in Syl_p(G)$. Then $P \neq G$. By (4.2), $P \neq N_G(P)$. If $N_G(H) \neq G$, then by (4.2) (with $H$ replaced by $N_G(H)$), $N_G(P) \neq N_G(N_G(H))$. On the other hand, by (4.3), $N_G(P) = N_G(N_G(H))$. This implies $N_G(H) = G$ and so $Syl_p(G) = \{P\}$.

**Step 2** If $P_i$ is a Sylow $p_i$-subgroup, and $P_j$ is a Sylow $p_j$-subgroup, where $p_i$ and $p_j$ are two distinct primes, then $P_i \cap P_j = \{e\}$ and $P_iP_j = P_iP_j$.

Consider the order of an element in the intersection to see $\{e\} = P_i \cap P_j$. For $x \in P_i, y \in P_j, xyx^{-1}y^{-1} \in P_iP_j$, as $P_i$ and $P_j$ are normal in $G$.

**Step 3** Suppose $|G| = p_1^{n_1}p_2^{n_2} \cdots p_m^{n_m}$, where $p_1, p_2, \ldots, p_m$ are the distinct primes dividing $|G|$, and $P_1, P_2, \ldots, P_m$ are the corresponding Sylow $p_i$-subgroups, then $G = P_1P_2 \cdots P_m$.

By Lagrange and Step 2, every element in $P_1 \cdots P_{i-1}P_{i+1} \cdots P_m$ has order dividing

$$p_1^{n_1}p_2^{n_2} \cdots p_{i-1}^{n_{i-1}}p_{i+1}^{n_{i+1}} \cdots p_m^{n_m},$$

and so

$$P_i \cap P_1 \cdots P_{i-1}P_{i+1} \cdots P_m = \{e\}.$$

As $P_1P_2 \cdots P_m \leq G$, and as

$$p_1^{n_1}p_2^{n_2} \cdots p_m^{n_m} = |P_1P_2 \cdots P_m| \leq |G| = p_1^{n_1}p_2^{n_2} \cdots p_m^{n_m},$$

we must have $G = P_1P_2 \cdots P_m$. 

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Combine Step 1 and Step 3 to conclude that

\[ G \cong P_1 \times P_2 \times \cdots \times P_m. \]

(7.5) (Thm 7.8) Let \( G \) be a group and let \( a, b \in G \). The element \( aba^{-1}b^{-1} \) is called a \textbf{commutator} of \( G \). The subgroup

\[ G' = \langle aba^{-1}b^{-1} : a, b \in G \rangle \]

is called the \textbf{commutator subgroup} of \( G \). Moreover,

(i) \( G' \) is an invariant subgroup of \( G \), and so \( G' \triangleleft G \).

(ii) \( G/G' \) is abelian.

(iii) Suppose \( N \triangleleft G \). Then \( G/N \) is abelian if and only if \( G' \triangleleft N \).

\textbf{Proof:} (i) \( \forall g \in G \), and \( \forall a, b \in G \),

\[ g(aba^{-1}b^{-1})g^{-1} = g(ab)g^{-1}(ab)^{-1}a(bg)^{-1}(bg)^{-1}gb^{-1}g^{-1}. \]

(ii) \( \forall a, b \in G, \ ab = aba^{-1}b^{-1}ba \), and so \( (ab)G' = (ba)G' \).

(iii) Suppose that \( (ab)N = (ba)N \iff aba^{-1}b^{-1} \in N \).

(7.6) Let \( G^{(0)} = G \). For each \( i \geq 1 \), define \( G^{(i)} = (G^{(i-1)})' \). Then

(i) \( \cdots \subseteq G^{(i)} \subseteq G^{(i-1)} \cdots \subseteq G^{(1)} \subseteq G^{(0)} = G \).

(ii) \( \forall i, G^{(i)} \triangleleft G \).

\textbf{Proof:} (7.5) \( \implies \) (ii) holds when \( i = 1 \). Then argue by induction and use the correspondence in the isomorphism theorems. (Another way to see this is that each of the \( G^{(i)} \) is an invariant subgroup.)

(7.7) A group \( G \) is \textbf{solvable} if \( G^{(n)} = \{e\} \) for some integer \( n \).

(7.8) Every nilpotent group is solvable.

\textbf{Proof:} \( G \) is nilpotent \( \implies C_n(G) = G \).

\( C(G/G_{n-1}(G)) = C_n(G)/C_{n-1}(G) \) is abelian \( \implies G^{(1)} \triangleleft C_{n-1}(G) \).

Similarly,

\[ G^{(2)} = (G^{(1)})' \triangleleft C_n(G) < C_{n-2}(G) \]

\[ G^{(n)} = (G^{(1)})' < C_1(G)' = C(G) = \{e\} \]

(7.9) (Thm 7.11)

(i) Every subgroup and every homomorphic image of a solvable group is solvable.

(ii) Suppose \( N \trianglelefteq G \). Then \( G \) is solvable if and only if both \( N \) and \( G/N \) are solvable.

**Proof:** (i). Suppose \( f : G \rightarrow H \) is an epimorphism. Then
\[
f(G^{(1)}) = \langle \{f(a)f(b)f(a)^{-1}f(b)^{-1} : a, b \in G\} \rangle = H^{(1)},
\]
and
\[
f(G^{(i)}) = \langle \{f(a)f(b)f(a)^{-1}f(b)^{-1} : a, b \in G^{(i-1)}\} \rangle = \langle \{f(a)f(b)f(a)^{-1}f(b)^{-1} : a, b \in H^{(i-1)}\} \rangle = H^{(i)}.
\]

If \( H < G \), then \( G^{(i)} \cap H = H^{(i)} \).

(ii). Let \( \pi : G \rightarrow G/N \) be the canonical map. Note that \( \pi(G^{(n)}) = (G/N)^{(n)} = \{e_{G/N}\} \). It follows that \( G^{(n)} \leq \ker \pi = N \). By (i), as a subgroup of a solvable group \( N \), \( G^{(n)} \) is also solvable.

(7.10) If \( n \geq 5 \), then \( S_n \) is not solvable.

**Proof** Suppose that \( S_n \) is solvable. Then by (4.9). \( A_n \), as a subgroup of \( S_n \), must be solvable. Consider the commutator group \( A'_n \). Since \( A_n \) is not abelian, \( A'_n \neq \{e\} \). Since \( A_n \) is simple, and since \( A'_n \triangleleft A_n \), we must have \( A'_n = A_n \). It follows that \( A^{(i)}_n = A_n \) for any \( i \geq 1 \), and so \( A_n \) cannot be solvable.

### A. Semidirect Products

(A.1) Let \( H \) and \( K \) be groups and let \( \phi : K \rightarrow Aut(H) \) be a homomorphism. (Notation: \( \phi(k)(h) = k \cdot h \).) Let \( G = \{(h, k) | h \in H \text{ and } k \in K\} \) with binary operation
\[
(h_1, k_1)(h_2, k_2) = (h_1(k_1 \cdot h_2), k_1k_2).
\]

Then each of the following holds:

(i) \( G \) with this operation forms a group, denoted by \( G = H \rtimes_\phi K \), called the **semidirect**
product of $H$ and $K$ with respect to $\phi$.

(ii) The sets $H' = \{(h, 1) | h \in H\}$ and $K' = \{(1, k) | k \in K\}$ are subgroups of $G$, isomorphic to $H$ and $K$, respectively.

(iii) $H' \leq G$.

(iv) $H' \cap K' = \{1\}$.

(v) $\forall h \in H$ and $k \in K$, $khk^{-1} = k \bullet h = \phi(k)(h)$.

**Proof** Check definitions.

(3.1a) When $\phi$ is the identity homomorphism, then

$$H \bowtie_{\phi} K \equiv H \times K.$$