Due Day: Oct. 16, 2006 Each problem is 10% of the exam. The extra credit problem counts an additional 20%. Message from the instructor: The extra credit problem requires a certain amount of analysis of the cases, and could be quite time consuming. Therefore, unless you have done all the regular problems with a high level of confidence, you are not recommended to work on the extra credit problem.

1. Let $H$ be a subgroup of a group $G$. Define (the centralizer of $H$ in $G$)

$$C_G(H) = \{g \in G : \forall h \in H, gh = hg\},$$

and (the normalizer of $H$ in $G$)

$$N_G(H) = \{g \in G : gH = Hg\}.$$

Show that $C_G(H)$ is a normal subgroup of $N_G(H)$.

Proof: We have shown earlier that $N_G(H)$ is a subgroup of $G$.

Firstly, we need to show that $C_G(H)$ is a subgroup of $N_G(H)$. Since $eh = he$ for any $h \in H$, $e \in C_G(H)$. If $a, b \in C_G(H)$, then $\forall h \in H$, $(ab)(h) = a(bh) = a(hb) = (ah)b = (ha)b = h(ab)$, and so $ab \in C_G(H)$. Since $a \in C_G(H)$, $\forall h \in H$, $ah = ha$. This means $ha^{-1} = a^{-1}aha^{-1} = a^{-1}haha^{-1} = a'^{-1}$, and so $a^{-1} \in C_G(H)$. Therefore $C_G(H) \leq G$.

Since $\forall g \in C_G(H)$, $\forall h \in H$, $gh = hg$, it follows that $gH = Hg$, $\forall g \in C_G(H)$, and so $C_G(H) \leq N_G(H)$, by the definition of $N_G(H)$.

Then, we apply the definition of normal subgroups to show that $C_G(H)$ is a normal subgroup of $N_G(H)$. For any $x \in N_G(H)$ and for any $a \in C_G(H)$, let $g = xax^{-1}$. It suffices to show that $g \in C_G(H)$. For any $h \in H$, $ghg^{-1} = xax^{-1}hxa^{-1}x^{-1}$. Since $x \in N_G(H)$, $h' = x^{-1}hx \in H$ (and so $xh'x^{-1} = h$). Since $a \in C_G(H)$, $ah' = h'a$. Thus

$$ghg^{-1} = xax^{-1}hxa^{-1}x^{-1} = xa(x^{-1}hx)a^{-1}x^{-1} = xah'a^{-1}x^{-1} = xha'a^{-1}x^{-1} = xh'(aa^{-1})x^{-1} = xhx^{-1} = h.$$

By the definition of $C_G(H)$, $g \in C_G(H)$, and so $C_G(H) \triangleleft N_G(H)$.

2. Let $G$ be a finite group with a prime $p > 1$ dividing $|G|$. Let $P$ be a Sylow $p$-subgroup of $G$. Show that $N_G(P) = N_G(N_G(P))$.

Proof: By the definition of a normalizer, any subgroup of $G$ is a subset of its normalizer in $G$. Thus $N_G(P) \subseteq N_G(N_G(P))$. It suffices to show that $N_G(N_G(P)) \subseteq N_G(P)$.

Let $H = N_G(N_G(P))$. Since $H$ is a subgroup of $G$, and since $P \subseteq N_G(P) \subseteq K$, $P$ is a Sylow $p$-subgroup of $H$. Then $\forall h \in H$, $hPh^{-1}$ is also a Sylow $p$-subgroup of $H$. Since $h \in H = N_G(N_G(P))$, by the definition of a normalizer, $h(N_G(P)h^{-1} = N_G(P)$. It follows $hPh^{-1} \subseteq h(N_G(P)h^{-1} = N_G(P)$, and so $hPh^{-1}$ is also a Sylow $p$-subgroup of $N_G(P)$. By Sylow’s 2nd Theorem, $\exists x \in N_G(P)$ such that $xP = hPh^{-1}$, But $P \subseteq N_G(P)$, $P = xP = hPh^{-1}$. It follows by the definition of a normalizer that $h \in N_G(P)$. Since $h \in H$ is arbitrary, $H \subseteq N_G(P)$,
as desired.

3. Show that a group of order 160 is not simple.

A proof imitating my lecture in class: Let $G$ be a group of order $160 = 5 \cdot 2^5$. Thus by Sylow’s 3rd Theorem, $n_2$, the number of Sylow 2-subgroups, is either 1 or 5. If $n_2 = 1$, then the only Sylow 2-subgroup of $G$ is normal in $G$. Thus we assume that $n_2 = 5$. Let $H_1, H_2, \cdots, H_5$ denote the Sylow 2-subgroups of $G$.

Let $N = H_i \cap H_j$, for some $i, j$ with $1 \leq i < j \leq 5$. Then by Lagrange, $|N|$ must be a factor of $|H_i| = 32$. Since

$$160 = |G| \geq |H_iH_j| = \frac{|H_i| \cdot |H_j|}{|N|} = \frac{1024}{|N|},$$

we have $|N| \in \{8, 16\}$. We consider two cases:

Case 1: For some $i, j$ with $1 \leq i < j \leq 5$, $N = H_i \cap H_j$ has order 16.

We may assume that $i = 1$ and $j = 2$. Then as $[H_1 : N] = 2$, $N \triangleleft H_1$, and so $H_1 \subseteq N_G(N)$. Similarly, $H_2 \subseteq N_G(N)$. It follows that

$$160 \geq |N_G(N)| \geq |H_1H_2| = \frac{|H_1| \cdot |H_2|}{|N|} = \frac{1024}{8} = 128.$$

Since $|N_G(N)|$ must be a factor of $|G| = 160$, it follows that $|N_G(N)| = 160$, and so $G = |N_G(N)|$. Therefore, $N \triangleleft G$, and $G$ is not simple.

Case 2: For any $i, j$ with $1 \leq i < j \leq 5$, $H_i \cap H_j$ has order 8.

Let $N = H_1 \cap H_2$. By Sylow’s first Theorem, there exist 2-subgroups $K_1$ and $K_2$, such that $N \leq K_1 \leq H_1$ and $N \leq K_2 \leq H_2$, and $|K_1| = |K_2| = 16$. Therefore, $[K_1 : N] = 2$, $N \triangleleft K_1$, and so $K_1 \subseteq N_G(N)$. Similarly, $K_2 \subseteq N_G(N)$.

$$160 \geq |N_G(N)| \geq |K_1K_2| = \frac{|K_1| \cdot |K_2|}{|N|} = \frac{256}{8} = 32.$$

Thus $|N_G(N)|$ is a factor of 160 and a multiple of 16. It follows that $|N_G(N)| \in \{32, 160\}$. If $|N_G(N)| = 160$, then $N_G(N) = G$, and $N \triangleleft G$. Therefore, we assume that $|N_G(N)|$ has order 32, and so it is a Sylow 2-subgroup $H_3$ (say).

Since $K_1 \leq H_3$, we have $[H_3 : K_1] = 2$. Thus $K_1 \triangleleft H_3$, and so $H_3 \subseteq N_G(K_1)$. Similarly, $H_1 \subseteq N_G(K_1)$. It follows that

$$160 \geq |N_G(K_1)| \geq |H_1H_3| = \frac{|H_1| \cdot |H_3|}{|H_1 \cap H_3|} = \frac{1024}{8} = 128.$$

Since $|N_G(K_1)|$ must be a factor of $|G| = 160$, it follows that $|N_G(K_1)| = 160$, and so $G = |N_G(K_1)|$. Therefore, $K_1 \triangleleft G$, and $G$ is not simple.

A proof using group actions: Let $H$ be a subgroup of $G$ and let $G$ acts on $G/H$, the set of left cosets of $H$ in $G$ by left multiplication. Then kernel of the action is a normal subgroup $N$ of $G$ contained in $H$. As a consequence, we have the Corollary 4.9 on page 91 of the textbook:

Corollary 4.9: If $H$ is a subgroup of $G$ with index $n$, and if $G$ does not have a nontrivial normal subgroup contained in $H$, then $G$ is isomorphic to a subgroup of $S_n$. 

\[ \text{ii} \]
Now let \(|G| = 160\) and let \(H\) be a Sylow 2-subgroup of \(G\). Then \(|G : H| = 5\). If \(G\) is simple, then by Corollary 4.9, \(G\) is isomorphic to a subgroup of \(S_5\). It follows that

\[160 = |G| \leq |S_5| = 5! = 120,\]

a contradiction. This indicates that \(G\) must have a nontrivial normal subgroup contained in \(H\).

4. Let \(p > 1\) be a prime. Show that any group \(G\) with \(|G| = p^2\) must be abelian.

**Proof:** Let \(C(G)\) be the center of \(G\). Since \(G\) is a \(p\)-group, by the class equation:

\[|C(G)| \equiv |G| \equiv 0 \pmod{p}.\]

Hence \(|C(G)| \neq 1\). By Lagrange, \(|C(G)|\) is a factor of \(|G| = p^2\), and so \(|C(G)| \in \{p, p^2\}\). If \(|C(G)| = p^2\), then \(G = C(G)\), and \(G\) is abelian. Suppose that \(|C(G)| = p\). Pick \(a \in G - C(G)\). Then \(C_G(a) = \{g \in G : ga = ag\} \leq G\) (you must verified this fact, or quote a former result). Then by the definition of these subgroups, \(C(G) \cup \{a\} \subseteq C_G(a)\), and so \(p = |C(G)| < |C_G(a)| \leq |G| = p^2\). By Lagrange, \(|C_G(a)| = p^2\) and so \(C_G(a) = G\). This implies that \(a \in C(G)\); contrary to the choice of \(a\). Hence we must have \(G = C(G)\), and so \(G\) must be abelian.

**A proof not using a former exercise:** Let \(C(G)\) be the center of \(G\). Since \(G\) is a \(p\)-group, by the class equation:

\[|C(G)| \equiv |G| \equiv 0 \pmod{p}.\]

Hence \(|C(G)| \neq 1\). By Lagrange, \(|C(G)|\) is a factor of \(|G| = p^2\), and so \(|C(G)| \in \{p, p^2\}\). If \(|C(G)| = p^2\), then \(G = C(G)\), and \(G\) is abelian. Suppose that \(|C(G)| = p\). Pick \(a \in G - C(G)\). Then \(C_G(a) = \{g \in G : ga = ag\} \leq G\) (you must verified this fact, or quote a former result). Then by the definition of these subgroups, \(C(G) \cup \{a\} \subseteq C_G(a)\), and so \(p = |C(G)| < |C_G(a)| \leq |G| = p^2\). By Lagrange, \(|C_G(a)| = p^2\) and so \(C_G(a) = G\). This implies that \(a \in C(G)\); contrary to the choice of \(a\). Hence we must have \(G = C(G)\), and so \(G\) must be abelian.

**A proof given the structure of \(G\):** If \(G\) is cyclic, then done, We assume that \(G\) is not a cyclic group. Therefore, by Lagrange and by Cauchy, every element of \(G\) other than the identity \(e\) must have order \(p\). Let \(a \in G - \{e\}\), and let \(H\) be the cyclic subgroup generated by \(a\). Pick an element \(b \in G - H\) and let \(K\) denote the subgroup generated by \(b\). Then we have these observations:

1. \(H \cap K = \{e\}\). In fact, if for some \(a^i \in H - \{e\}\) and some \(b^j \in K - \{e\}\), \(a_i = b^j\), then since \(p\) is a prime, \(H = \langle a^i \rangle = \langle b^j \rangle = K\), contrary to the choice of \(b\).
2. By Sylow’s 1st Theorem, both \(H \triangleleft G\) and \(K \triangleleft G\).
3. By counting,

\[p^2 = |G| \geq |HK| = \frac{|H| \cdot |K|}{|H \cap K|} = p^2,
\]

and so \(G = HK\). It follows by (1), (2) and (3) that \(G = H \oplus K \cong \mathbb{Z}_p \times \mathbb{Z}_p\) is an abelian group.

5. Show that every group of order 15 must be abelian.

**Proof:** Let \(G\) be a group of order 15. Then by Sylow, \(G\) has a normal subgroup \(H\) of order 3, and a normal subgroup \(K\) of order 5. We have the following observations:

1. Since 3 and 5 are relatively primes, by Lagrange, \(H \cap K = \{e\}\).
2. \(H \triangleleft G\) and \(K \triangleleft G\) (by Sylow’s 3rd Theorem, stated above).
(3) By counting,
\[ 15 = |G| \geq |HK| = \frac{|H| \cdot |K|}{|H \cap K|} = 15. \]
and so \( G = HK \). It follows by (1), (2) and (3) that \( G = H \oplus K \cong \mathbb{Z}_p \times \mathbb{Z}_p \) is an abelian group.

**Extra Credit Problem:** (No partial credit will be given to a solution of this extra credit problem unless the presented solution is part of a correct solution).

(A) Wooden cubes of the same size are to be painted a different color on each face to make children’s blocks. How many distinguishable blocks can be made if 8 colors or paint are available?

(B) What the answer to (A) will be if colors may be repeated on different faces?

(Hint: Let \( S \) be the set of all possible colored cubes. Find a group acting on \( S \) so that each orbit represents precisely one distinguishable block. Then count the number of orbits using the class equation).

Solution: Let 1 and 6 denote the top and the bottom of the cube, and let 2, 3, 4, 5 label the lateral faces of the cube. The group of rigid motion of the cube is the subgroup \( G \) of \( S_6 \) generated by \( a = (1 \ 4 \ 6 \ 2), b = (1 \ 3 \ 6 \ 5), c = (2 \ 3 \ 4 \ 5) \) (and so the elements in the cyclic subgroups generated by \( a, b \) and \( c \) are rotations fixing a pair of opposite faces). (Note that \( ab = (1 \ 3 \ 2)(4 \ 6 \ 5), ba = (1 \ 4 \ 5)(2 \ 3 \ 6) = ac = cb, ca = (1 \ 5 \ 2)(3 \ 4 \ 6), bc = (1 \ 3 \ 4)(2 \ 6 \ 5), \) which represent some of the rotations fixing a pair of opposite vertices; and \( a^2b = (1 \ 3)(2 \ 4)(5 \ 6), \ a^2c = (1 \ 6)(2 \ 3)(4 \ 5), \) which represent some of the rotations fixing a pair of opposite edges). Note that \( G \) has 24 elements. The consists of the identity, 9 rotations that leave a pair of opposite faces unchanged, 8 that leave a pair of opposite vertices unchanged, and 6 that leave a pair of opposite edges unchanged.

(A) The total number of ways to color this cube is \( 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \). The only element in \( G \) that leaves unchanged a cube with different colors on all the faces is the identity. By Burnside’s Lemma, The number of distinguishable cubes is
\[ \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{24} = 840. \]

(B) In this case, there are \( 8^6 \) ways of coloring the faces of a cube. We use Burnside’s Lemma to count the number of orbits. Let \( \iota \) denote the identity of the group \( G \).

\[
\begin{align*}
|X_\iota| &= 8^6 \\
|X_{\text{opp. face, 90}^\circ \text{ rotation}}| &= 8^3 \text{ (with 6 of this kind)} \\
|X_{\text{opp. face, 180}^\circ \text{ rotation}}| &= 8^4 \text{ (with 3 of this kind)} \\
|X_{\text{opp. vertices}}| &= 8^2 \text{ (with 8 of this kind)} \\
|X_{\text{opp. edgss}}| &= 8^3 \text{ (with 6 of this kind)}
\end{align*}
\]

Thus by Burnside’s Lemma, the number of orbits (distinguishable colored cubes) is
\[
\frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{8^6 + 6 \cdot 8^3 + 3 \cdot 8^4 + 8 \cdot 8^2 + 6 \cdot 8^3}{24} = 11712.
\]