Exercises on Compact and Connected Subsets in $\mathbb{R}^n$

(2.28) Show that compactness is a topological property and give examples to show that closedness and boundedness are not.

**Proof:** Suppose that $X$ and $Y$ are topological spaces, and $f : X \mapsto Y$ is a homeomorphism. That is, $f : X \mapsto Y$ is a bijection, $f$ and $f^{-1} : Y \mapsto X$ are continuous.

(i) Compactness is a topological property.

This is to show that if $X$ is compact, then $Y$ is also. Suppose that $X$ is compact. We want to show that $Y$ is also compact. Our strategy is to apply the definition, and show that every infinite sequence of points in $Y$ has a limit point in $Y$.

Let $\{y_i\}_{i=1}^{\infty}$ be an infinite sequence of points in $Y$. Let $x_i = f^{-1}(y_i)$, $i = 1, 2, ...$. Then $\{x_i\}_{i=1}^{\infty}$ is an infinite sequence of points in $X$. Since $X$ is compact, $\{x_i\}_{i=1}^{\infty}$ has a limit point $a$ in $X$. Since $f : X \mapsto Y$ is continuous, by Theorem (2.16) and Theorem (2.12), $f(a)$ is a limit point of $\{f(x_i)\}_{i=1}^{\infty} = \{y_i\}_{i=1}^{\infty}$. Since $f(a) \in Y$, the sequence $\{y_i\}_{i=1}^{\infty}$ has a limit point in $Y$. By the definition of compact sets, $Y$ is compact.

(ii) closedness and boundedness are not topological properties.

Let $f(x) = \tan(x)$, $X = (-\frac{\pi}{2}, \frac{\pi}{2}) \subseteq \mathbb{R}$ and $Y = (-\infty, \infty) = \mathbb{R}$. Then $f : X \mapsto Y$ is a homeomorphism (need to check this fact). $X$ is bounded but $Y$ is not bounded; and $X$ is not closed in $\mathbb{R}$ but $Y$ is closed in $\mathbb{R}$.

(2.29) Prove that $X$ is connected if and only if $X$ cannot be written as a union of two non-empty disjoint sets which are closed related to $X$.

**Proof:** First we assume that $X$ is connected. By contradiction, we assume that $X = A \cup B$ such that $A \neq \emptyset$ and $B \neq \emptyset$, $A \cap B = \emptyset$ and $A$ and $B$ are closed related to $X$. (So we are looking for a contradiction).

Since $X$ is connected, by the definition of connected sets, either $A$ or $B$ contains a limit point of the other. We may assume that $A$ contains a limit point $b$ of $B$. Since $B$ is closed, and since $b \in A = A - B$, $b$ has a neighborhood $N$ such that $b \in N$ and $N \cap B = \emptyset$. By the definition of limit points, $b$ cannot be a limit point of $B$, contrary to the assumption that $b$ is a limit point of $B$. This contradiction shows that $X$ cannot be written as a union of two non-empty disjoint sets which are closed related to $X$.

Conversely, we assume that $X$ cannot be written as a union of two non-empty disjoint sets which are closed related to $X$. We will verify the definition of connected sets for $X$. Suppose that for some subsets $A$ and $B$ of $X$,

$$X = A \cup B, A \neq \emptyset, B \neq \emptyset, A \cap B = \emptyset.$$

By the assumption, $A$ and $B$ cannot be both closed in $X$. We assume that $A$ is not closed in $X$. 

Then by Exercise 2.5 (done before), \(A\) cannot be equal to its closure \(\text{Cl}(A)\). Thus there must be a point \(a \in \text{Cl}(A) - A\), or \(a \in \text{Fr}(A)\) by the definition of \(\text{Cl}(A)\). Pick one such \(a \in \text{Cl}(A) - A\). Then \(A\) has a limit point (frontier point) \(a\) which is not in \(A\). Since \(X = A \cup B\) and since \(a \in X - A\), we have \(a \in B\), and so \(B\) contains a limit point of \(A\). By the definition of connected sets, \(X\) is connected.