Exercises on Open and Closed Sets in $\mathbb{R}^n$

(2.1) Determine whether the following subsets of $\mathbb{R}^2$ are open, closed, and/or bounded.

(1) $A = \{ \|x\| \leq 1 \}$.
(2) $B = \{ \|x\| = 1 \}$.
(3) $C = \{ \|x\| < 1 \}$.
(4) $D = \{ \text{the x-axis} \}$.
(5) $E = \mathbb{R}^2 - \{ \text{the x-axis} \}$.
(6) $F = \{ (x, y) : x \text{ and } y \text{ are integers} \}$.
(7) $G = \{ (1, 0), (\frac{1}{2}, 0), (\frac{1}{3}, 0), \cdots, (\frac{1}{n}, 0), \cdots \}$.
(8) $H = \mathbb{R}^2$.
(9) $I = \emptyset$.

Solution: (Key to get correct answers: Check the definitions).

(1) For any $x \in A$, we have $\|x\| \leq 1$ and so by the definition of bounded sets, $A$ is bounded.

For any $y \in \mathbb{R}^2 - A$, we have $\|y\| > 1$ (by the definition of $A$), Let $r = (\|y\| - 1)/2$. We are to check if $D^2(y, r) \cap A = \emptyset$. Argue by contradiction to show this. Suppose that $D^2(y, r) \cap A \neq \emptyset$ and so $\exists z \in D^2(y, r) \cap A$. Then by the property of distances (triangular inequality of distance: $\|a + b\| \leq \|a\| + \|b\|$),

$$\|y\| = \|y - 0\| \leq \|y - z\| + \|z - 0\| \quad \text{by triangular inequality}$$
$$< r + 1 \quad \text{since } z \in D^2(y, r) \text{ and } z \in A$$
$$\leq \|y\| + 1 \quad \text{algebraic manipulations}$$
$$< \|y\| \quad \text{because } r = (\|y\| - 1)/2$$

and so a contradiction obtains. This indicates that $D^2(y, r) \cap A = \emptyset$ and so by definition of closed sets, $A$ is closed.

(2) $B = \{ \|x\| = 1 \}$ is closed. Sketch of Solution: \forall y \in \mathbb{R}^2 - B, let

$$r = \begin{cases} 
(\|y\| - 1)/2 & \text{if } \|y\| > 1 \\
(1 - \|y\|)/2 & \text{if } \|y\| < 1 
\end{cases}$$

Then imitate the argument in (1) to show that $D^2(y, r) \cap B = \emptyset$.

(3) $C = \{ \|x\| < 1 \}$ is open. Sketch of Solution: \forall y \in C, let $r = (1 - \|y\|)/2$. Then imitate the argument in (1) to show that $D^2(y, r) \subseteq C$.

(4) $D = \{ \text{the x-axis} \}$ is closed. Sketch of Solution: \forall y = (y_1, y_2) \in \mathbb{R}^2 - D, by the def. of $D$, $y_2 \neq 0$. Let $r = |y_2|/2$. Then explain why $D^2(y, r) \cap D = \emptyset$. (Draw a picture to help yourself).

(5) $E = \mathbb{R}^2 - \{ \text{the x-axis} \} = \mathbb{R}^2 - D$ is open. We can either use definition of open sets and
imitate the arguments above, or we can prove Exercise (2.11) (which is basically done in class) and apply the conclusion of (4) that $D$ is closed.

(6) $F = \{(x, y): x$ and $y$ are integers $\}$ is closed. Sketch of Solution: (Draw a picture to help yourself). Let $z = (z_1, z_2) \in \mathbb{R}^2 - F$. Then $z$ must be lying in a unit length square each of whose vertices are in $F$. We may denote $m = [x]$ and $n = [y]$, and so the four vertices of this unit square containing $z$ are $(m, n), (m + 1, n), (m, n + 1), (m + 1, n + 1)$. Let $r_1$ denote the smallest distance between $z$ and these four vertices, and let $r = r_2/2$. Then explain why $D^2(z, r) \cap F = \emptyset$.

(7) $G = \{(1, 0), (\frac{1}{2}, 0), (\frac{1}{3}, 0), \ldots, (\frac{1}{m}, 0), \ldots\}$ is not open nor closed. Sketch of Solution: (Draw a picture to help yourself). For any $r > 0$, then neighborhood centered at $(1, 0)$ contains a point $(1, r/2)$ which is not in $G$, and so $G$ has points which are not interior in $G$, and so by the definition of open sets, (Definition (2.3)), $G$ is not open.

For any $r > 0$, then neighborhood centered at $(0, 0)$ with radius $r$ contains all the points $(\frac{1}{n}, 0)$ whenever $n > 1/r$. Therefore, $(0, 0)$ is a limit point of $G$ but $(0, 0)$ is not in $G$. In other words, $(0, 0)$ is not exterior of $G$, and so $G$ is not closed, by the definition of closed sets (Def (2.2)(2)).

(7A) (For comparison, we show that by adding one point to $G$, we can get a closed set). $G' = \{(0, 0), (1, 0), (\frac{1}{2}, 0), (\frac{1}{3}, 0), (\frac{1}{4}, 0), \ldots, (\frac{1}{m}, 0), \ldots\}$ is closed. Sketch of Solution: (Draw a picture to help yourself). Let $z = (z_1, z_2) \in \mathbb{R}^2 - G'$. By the def. of $G'$, either $z_2 \neq 0$ or $z_2 = 0$ but $z_1 \notin \{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}$. When $z_2 = 0$ and $0 < z_1 \notin \{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}$, there must be an integer $N$ such that $\frac{1}{N+1} < z_1 < \frac{1}{N}$. Let

$$r = \begin{cases} |z_2|/2 & \text{if } z_2 \neq 0 \\ \min\{z_1 - \frac{1}{N+1}, \frac{1}{N} - z_1\}/2 & \text{if } z_2 = 0 \text{ and } \frac{1}{N+1} < z_1 < \frac{1}{N} \end{cases}$$

Then imitate the argument in (4) to show that $D^2(z, r) \cap G' = \emptyset$. The argument for the case when $z_2 = 0$ and $z_1 < 0$ is similar, in which case we can use $r = |z_1|/2$.

(8) $E = \mathbb{R}^2$ is both open and closed. To see that $E$ is open, just verify the definition, which is straightforward as $X$ contains all subsets, no matter they are neighborhood of a point or not. To see $E$ is closed, simply notice that $\mathbb{R}^2 - E = \emptyset$, and so the def. of closed sets is vacuously true.

(9) $I = \emptyset$ is both open and closed. (Imitate the argument in (8)).

(2.2) Show that $Cl(A) = \{\text{limit points of } A\}$.

Proof: (Before we start working, we ask ourselves the questions what is $Cl(A)$ and what are limit points of $A$.) We refresh our minds by looking at these definitions.

(1) Limit points of $A = \{x \in \mathbb{R}^n : \text{every neighborhood of } x \text{ contains at least one pt in } A\}$

(2) $Cl(A) = A \cup Fr(A)$.

(3) $Fr(A) = \{x \in \mathbb{R}^n : \text{every neighborhood of } x \text{ contains at least one pt in } A \text{ and one point} \}$
Let $B = \{ \text{limit points of } A \}$. We are to show that $Cl(A) = B$. (How do we show two sets are the same?) We shall show that $Cl(A) \subseteq B$ and $Cl(A) \supseteq B$.

Let $x \in Cl(A) = A \cup Fr(A)$. If $x \in A$, then every neighborhood of $x$ contains at least one pt (x itself) in $A$, if $x \in Fr(A)$, then by the definition of $Fr(A)$, every neighborhood of $x$ contains at least one pt in $A$. Therefore, by (1) (the definition of limit points of $A$), $x \in B$. This proves $Cl(A) \subseteq B$.

Let $x \in B$. If $x \in A$, then by (2) (the definition of $Cl(A)$), $x \in Cl(A)$. Now assume that $x \not\in A$. By (1) (the definition of limit points of $A$), every neighborhood of $x$ contains at least one pt in $A$, and so by (3) (definition of $Fr(A)$), $x \in Fr(A) \subseteq Cl(A)$. Thus in any case, $x \in Cl(A)$, and so $Cl(A) \supseteq B$.

Comments: These two examples show us how to proceed a proof type of exercise. We will be making several claims/conclusions, which eventually lead us to the desired conclusion that completes the exercise. Each claim requires a reason. The only things that can be accepted as a legal reason are: definitions, known results (such as results that have been proved in class, or in Calculus or geometry, such as the triangular inequality of distance), correct algebraic manipulations. As the exercises will train us to have the correct way to present ourselves, any claim/conclusion made without correct reasons will lead to point deductions.

(2.5) A set $A$ is closed if and only if $A = Cl(A)$.

Proof: If $A = Cl(A)$, then by the Theorem that $Cl(A)$ is closed, $A$ is closed. (See we can use a theorem shown in class as a sound reason).

We assume that $A$ is closed to prove that $A = Cl(A)$.

Since $Cl(A) = A \cup Fr(A)$, we have $A \subseteq Cl(A)$. To show that $Cl(A) \subseteq A$, we shall show that every point $y \not\in A$ must also be not in $Cl(A)$.

Suppose that $y \not\in A$. By the definition of a closed set, $y$ has a neighborhood $N$ such that $N \cap A = \emptyset$. By the definition of $Fr(A)$, $y \not\in Fr(A)$ also. Hence $y \not\in A \cup Fr(A) = Cl(A)$.

(2.6) For any set $A$, $Fr(A)$ is closed.

Proof: By the definition of $Fr(A)$, if $x \in Fr(A)$, then any neighborhood $N$ of $x$ contains points in $A$ and points not in $A$. Therefore, any point $y \not\in Fr(A)$ has one neighborhood $N$ such that either $N \cap A = \emptyset$ or $N \subseteq A$.

We shall show that $N \cap Fr(A) = \emptyset$, and so by the definition of closed sets, $Fr(A)$ is closed.

Suppose that $z \in N \cap Fr(A)$. Then by the definition of $Fr(A)$, the neighborhood $N$ of $z$ must satisfy $N \cap A \neq \emptyset$. By the choice of $N$, we must have $N \subseteq A$, and so it is impossible for
to contain a point not in \( A \), contrary to the assumption that \( z \in Fr(A) \). This contradiction indicates that \( N \cap Fr(A) = \emptyset \), and so by the definition of closed sets, \( Fr(A) \) is closed.

(2.9) A set \( A \) is open if and only if \( A = Int(A) \).

Proof: By the definition of \( Int(A) \), a point \( x \in Int(A) \) is \( x \) has a neighborhood \( N \) such that \( N \subseteq A \), and so \( Int(A) \subseteq A \).

We need to show that \( A \subseteq Int(A) \). \( \forall x \in A \), since \( A \) is open, \( x \) has a neighborhood \( N \) such that \( N \subseteq A \). By the definition of open sets, \( x \in Int(A) \). Thus \( A \subseteq Int(A) \).

(2.13) If \( A \) and \( B \) are open sets, prove that \( A \cup B \) and \( A \cap B \) are open. Give an example of a sequence of open sets \( A_1, A_2, \cdots \), such that \( \bigcap_{i=1}^{\infty} A_i \) is not open.

Proof: Let \( X = A \cup B \) (or \( X = A \cap B \), respectively). It suffices to show that \( X = Int(X) \), or every point \( X \) is an interior point of \( X \). Let \( x \in X \) be arbitrarily chosen. Then \( x \in A \) or \( x \in B \) (\( x \in A \) and \( x \in B \), respectively). Since \( A \) and \( B \) are open, \( x \) has a neighborhood \( D^n(x, r_1) \subseteq A \) and a neighborhood \( D^n(x, r_2) \subseteq B \).

If \( X \subseteq A \cup B \), then \( x \) has a neighborhood \( D^n(x, r_1) \subseteq A \subseteq A \cup B \), and so \( x \in Int(X) \). By the definition of open sets, \( X = A \cup B \) is open.

Now suppose that \( X = A \cap B \). Choose \( r = \min\{r_1, r_2\} \). Then \( x \) has a neighborhood \( D^n(x, r) \). Since \( r \leq r_1 \), \( D^n(x, r) \subseteq A \); since \( r \leq r_2 \), \( D^n(x, r) \subseteq B \). It follows that \( x \) has a neighborhood \( D^n(x, r) \subseteq A \cap B \). By the definition of open sets, \( X = A \cap B \) is open.

To find an example of a sequence of open sets \( A_1, A_2, \cdots \), such that \( \bigcap_{i=1}^{\infty} A_i \) is not open, we let \( A_n = (-1/n, 1/n) \), for \( n = 1, 2, \cdots \). Then since each \( A_n \) is an open interval, it is shown in class (or by Exercise 2.1 (1) with \( R^n = R \)) that \( A_n \) is open in \( R \). The intersection \( \bigcap_{n=1}^{\infty} = \{0\} \), is not an open set in \( R \).