Math 156 Fall 2012  Quiz 9 Solutions

Name:

Instruction. Appropriately use Divergency Test, Integral Test, Comparison Test and Limiting Comparison Test to determine if each of the given series is convergent or divergent. For each problem, you must indicate (A) which test is applied, and (B) how the test is applied to claim your conclusion. Failing to do so would result in point deductions.

1: \[ \sum_{n=1}^{\infty} \frac{2n + 1}{5n - 1}. \]

Solution: As discussed in class, we first apply the Divergency Test and compute (we can apply L’Hôspital’s Rule once)

\[
\lim_{n \to \infty} \frac{2n + 1}{5n - 1} = \lim_{n \to \infty} \frac{2}{5} = \frac{2}{5} \neq 0.
\]

By the divergency Test, \( \sum_{n=1}^{\infty} \frac{2n + 1}{5n - 1} \) diverges.

2: \[ \sum_{n=3}^{\infty} \frac{1}{n(\ln(n))^3}. \]

Solution: As commented in class, series with general term having the form \( \sum_{n=3}^{\infty} \frac{1}{n(\ln(n))^k} \)

wold be most effectively dealt with by Integral Test.

Again, we start with computing

\[
\lim_{n \to \infty} \frac{1}{n(\ln(n))^3} = 0,
\]

as the denominator goes to infinity. Therefore, the Divergency Test does not apply.

Choose \( f(x) = \frac{1}{x(\ln(x))^3} \). Then \( a_n = f(n) \). By chain rule and product rule, \( f'(x) = \frac{-1}{(x(\ln(x))^3)^2} \cdot \left( (\ln(x))^3 + \frac{x \cdot 3(\ln(x))^2}{x} \right) < 0 \) when \( x \geq 3 \). Therefore, \( f(x) \) is decreasing.

Now compute the integral. Set \( u = \ln(x) \) in the following integral. Then \( du = \frac{1}{x}dx \) and so

\[
\int \frac{1}{x(\ln(x))^3}dx = \int \frac{du}{u^3} = \frac{-1}{2u^2} = -\frac{1}{2(\ln(x))^2}.
\]

Thus

\[
\int_{3}^{\infty} \frac{1}{x(\ln(x))^3}dx = \lim_{A \to \infty} -\frac{1}{2(\ln(x))^2}\bigg|_{3}^{\infty} = \lim_{A \to \infty} \left( \frac{1}{2(\ln(3))^2} - \frac{1}{2(\ln(A))^2} \right) = \frac{1}{2(\ln(3))^2} - 0.
\]
Therefore, the integral is convergent. By the **Integral Test**, \( \sum_{n=3}^{\infty} \frac{1}{n(\ln(n))^3} \) converges.

3: \( \sum_{n=2}^{\infty} \frac{2}{n^3 + 4} \). Here \( a_n = \frac{2}{n^3 + 4} \).

**Solution:** Again, we start with computing

\[
\lim_{n \to \infty} \frac{2}{n^3 + 4} = 0,
\]
as the denominator goes to infinity. Therefore, the **Divergency Test** does not apply.

An observation on \( a_n \) shows that it is basically a \( p \)-series with \( p = 3 \). If we use **Comparison Test**, we set \( b_n = \frac{2}{n^3} \). Since the denominator of \( a_n \) is bigger than that of \( b_n \), we have

\[
a_n = \frac{2}{n^3 + 4} \leq \frac{2}{n^3} = b_n.
\]

Since \( \sum_{n=2}^{\infty} \frac{2}{n^3} = 2 \sum_{n=2}^{\infty} \frac{1}{n^3} \) is a \( p \)-series with \( p = 3 > 1 \), it is convergent. By the **Comparison Test**, \( \sum_{n=2}^{\infty} \frac{2}{n^3 + 4} \) converges.

**Another Solution.** We can also use **Limiting Comparison Test**. Same observation leads us to set \( b_n = \frac{1}{n^3} \). Instead of comparing which one is bigger, we computing the limit (we may apply L’Hospitail’s Rule 3 times or use algebraic manipulations):

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{2n^3}{n^3 + 4} = 2.
\]

Since \( \sum_{n=2}^{\infty} \frac{1}{n^3} \) is a \( p \)-series with \( p = 3 > 1 \), it is convergent. As \( 0 < 2 < \infty \), by the **Limiting Comparison Test**, \( \sum_{n=2}^{\infty} \frac{2}{n^3 + 4} \) converges.

**Grade distribution of this quiz** (including the worksheet extra credit):

**Meaning of the scores:**

\( \geq 10 = \text{Very good} \), familiar with the related materials and skillful, with minimal computational errors. Keep on!

\( 9 = \text{good} \), familiar with most of the related materials, with a few computations errors. Make an effort to do better.
Scores

<table>
<thead>
<tr>
<th>Scores</th>
<th>≥ 10</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>≤ 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>6</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>Percentage</td>
<td>18.2</td>
<td>12.1</td>
<td>15.1</td>
<td>9</td>
<td>3</td>
<td>24.4</td>
<td>18.2</td>
</tr>
</tbody>
</table>

Discussions and Comments

(1) **Problem:** Differentiation seems to be a major problem for most of us. Here are the facts observed in the work sheet and quiz during this period:

1. No one student computed the derivative of $2^x$ and $3^x$ correctly (which was needed in using L’Hospital’s Rule).

2. Among the 33 students taking the quiz this week, only two (out of 33) knows how to compute the derivative $\frac{1}{x(\ln(x))^3}$ correctly. (Both Chain Rule and Quotient Rule are needed).

3. Several of us perform the following differentiation $\left(\frac{x^2}{x^3 + 1}\right)' = \frac{2x}{3x^2}$. (This is the right moment to apply quotient rule or product rule).

**What to do:** As differentiation is one of the most important concept and skill in Calculus, failing to do it right will cause our failure in subsequent math courses. I once again urge everyone to be familiar with the differentiation formulas, Formula 1-24 on Page 5 of the reference pages in the text. You will need them in both Exam 3 and the final exam.

(2) **Problem:** Many of us also seem to have trouble in computing limits. Here are some common errors observed (at least 5 student made such mistakes for each error listed below):

\[
\lim_{n \to \infty} \frac{n}{n} = \infty
\]

\[
\lim_{n \to \infty} \frac{4 + 3^n}{2^n} = \infty = 1
\]
$$\lim_{n \to \infty} \frac{1}{\ln(n)} = \infty$$  \hspace{1cm} (more than 1/3 of us made this and similar mistakes)

$$\lim_{n \to \infty} \frac{2n + 1}{5n - 1} = 0$$

I left for you to find the correct answer for each.

**What to do:** We might have the feeling now that when series are on the stage, we would need almost every thing we have learned in Calculus 1 to perform well. The skill of limit computing is also one of them. It is time for us to refresh our limit computing skills, as it will be needed in both Exam 3 and in the Final Exam.

**(3) Problem:** Algebraic errors seems to be sticky to some of us. This time, will be the justification of inequalities. The following are some examples I found more than 6 students did each of them.

\[
\frac{n^2}{n^3 + 1} \geq \frac{1}{n} \quad \text{Correct Version:} \quad \frac{n^2}{n^3 + 1} < \frac{1}{n}.
\]

\[
\frac{2n + 1}{5n - 1} \geq \frac{2n}{5n} \left( \frac{2}{5} \right)^n \quad \text{Try } n = 1 \text{ to see the absurdity.}
\]

\[
\frac{n^3}{n^3 + 1} = \frac{n^3}{1 + \frac{n^3}{n^3}} \quad \text{Not possible even in Alice’s Wonderland. Just try } n = 1 \text{ to see the absurd.}
\]

**What to do:** We should train ourselves to do accurate computation when we are working on our homework. Then such errors will be avoided.

**(4) Problem:** Those they attend the class regularly might have remember that I said "Divergency Test" Can never be used to conclude convergency. Many in the quiz did. I classify this group of errors as "creating our own erroneous tests". Here are the common errors in this group.

1. 9 out of 33 students computed \( \lim_{n \to \infty} a_n \) and concluded "the series is convergent", when it is either zero or the limit exists.

2. 11 students performed the comparison \( a_n \leq \frac{1}{n} \) and then claimed that \( \sum a_n \) diverges.

3. 7 or us considered \( \sum \frac{1}{(\ln(n))^3} \) is a \( p \)-series with \( p = 3 \). This is a misunderstanding of \( p \)-series. Please read the summary again to see what \( p \)-series are only those of the form \( \sum \frac{1}{n^p} \).

**What to do:** The tools in our tool box are for us to use. As the summary sheet has all such convergency tests, we should follow the tests and use them correctly. Once again, I recommend that we only use geometric series and \( p \)-series as our comparison standards.