Math 156 Fall 2012  Quiz 6

Name:

1: Determine whether the integral \( \int_{0}^{\infty} xe^{-x^2} \, dx \) is convergent or divergent. Evaluate it if convergent.

**Solution:** This is an improper integral over an infinite interval. We first convert it into a finite interval proper integral, then take the appropriate limit to find its value or conclude that is divergent. The indefinite integral can be computed as follows: \( u = x^2, \, du = 2x \, dx \)

\[
\int xe^{-x^2} \, dx = \frac{1}{2} \int e^{-u} \, du = \frac{-e^{-u}}{2} = \frac{-e^{-x^2}}{2}.
\]

Thus

\[
\int_{0}^{\infty} xe^{-x^2} \, dx = \lim_{A \to \infty} \int_{0}^{A} xe^{-x^2} \, dx = \lim_{A \to \infty} \left[ -\frac{e^{-x^2}}{2} \right]_{0}^{A} = \lim_{A \to \infty} \left( \frac{1}{2} - \frac{1}{2e^{A^2}} \right) = \frac{1}{2} - 0 = \frac{1}{2}.
\]

The integral is convergent and its value equals \( \frac{1}{2} \).

2: Determine whether the integral \( \int_{1}^{5} \frac{1}{\sqrt{x-1}} \, dx \) is convergent or divergent. Evaluate it if convergent.

**Solution:** This is an improper integral with a discontinuity at \( x = 1 \). We first convert it into a proper integral with a continuous integrand, then take the appropriate limit to find its value or conclude that is divergent. The indefinite integral can be computed as follows: \( u = x - 1, \, du = dx \)

\[
\int \frac{1}{\sqrt{x-1}} \, dx = \int u^{-\frac{1}{2}} \, du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} = 2u^{\frac{1}{2}} = 2\sqrt{x-1}.
\]

Thus

\[
\int_{1}^{5} \frac{1}{\sqrt{x-1}} \, dx = \lim_{d \to 0^+} \int_{1+d}^{5} \frac{1}{\sqrt{x-1}} \, dx = \lim_{d \to 0^+} \left[ 2\sqrt{x-1} \right]_{1+d}^{5} = \lim_{d \to 0^+} \left( 2\sqrt{5-1} - 2\sqrt{(1+d)-1} \right) = 2\sqrt{4} - 0 = 4.
\]

The integral is convergent and its value equals 4.

3: Determine whether the integral \( \int_{0}^{1} \frac{3}{x^5} \, dx \) is convergent or divergent. Evaluate it if convergent.
Solution: This is an improper integral with a discontinuity at $x = 0$. We first convert it into a proper integral with a continuous integrand, then take the appropriate limit to find its value or conclude that is divergent. The indefinite integral can be computed as follows:

$$\int \frac{3}{x^5} \, dx = \int 3x^{-5} \, dx = \frac{3x^{-5+1}}{-5+1} = \frac{3}{-4x^4}.$$ 

Thus,

$$\int_0^1 \frac{3}{x^5} \, dx = \lim_{d \to 0^+} \int_d^1 \frac{3}{x^5} \, dx = \lim_{d \to 0^+} \left[ \frac{3}{-4x^4} \right]_d^1 = \lim_{d \to 0^+} \left( \frac{3}{-4} - \frac{3}{-4d^4} \right),$$

and the limit does not exists. The integral is divergent.

Grade Distribution of this quiz (including the worksheet extra credit):

Meaning of the scores:

$\geq 10 =$ Very good, familiar with the related materials and skillful, with minimal computational errors. Keep on!

$9 =$ good, familiar with most of the related materials, with a few computations errors. Make an effort to do better.

$8 =$ OK, not so familiar with the related materials, with relatively more computational errors. We have room to improve. (For this quiz, not familiar with differentiation).

$7 =$ Passing, less familiar with the related materials and more computational errors and algebraic errors. We have lots of room to improve.

$5, 6 =$ Borderline. We need to catch it up. If you have trouble doing your homework, it might be time to visit your instructor to get help. Do not wait to let the trouble accumulate.

Below 4 =$ This might be a dangerous warning signal. We are failing! It should definitely be the time for us to see the instructor and get assistance to understand the materials and to practice MORE.

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<th>7</th>
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Discussions and Comments

It is a very good phenomena that most of us understand how to deal with improper integrals by converting them into proper ones together with the appropriate limiting
(1) **Problem:** The following are two major type of limit errors:

Erroneous solution: \( \lim_{A \to \infty} e^{-\frac{A^2}{2}} = \infty \)  
Correct Solution: \( \lim_{A \to \infty} e^{A^2} = \infty \).

Erroneous solution: \( \lim_{d \to 0^+} \frac{3d^4}{4} = \frac{3 \cdot 0^4}{4} = 0 \)  
Correct Solution: \( \lim_{d \to 0^+} \frac{3}{4d^4} = \infty \).

**What to do:** Some of us are in fact correctly sensed something by writing \( \lim_{A \to \infty} \frac{e^{-A^2}}{2} = \frac{1}{\infty} = 0 \), and by writing \( \lim_{d \to 0^+} \frac{3d^4}{4} = \frac{3}{0} = 0 \). We would need to **understand** what limit means instead of remembering what they are. Some others seemed to have trouble dealing with negative exponents, which might be an algebra weakness.

(2) **Problem:** A few of us, especially those who scored low, continue to show weakness in algebra.

**What to do:** It is good that many are making progress in strengthen our algebraic skills. Practice more seems to be the most effective prescription for making our algebra strong.