Math 156 Fall 2012  Quiz 5 Solutions

Name:

The procedure of finding partial fraction decompositions of fractional functions \( \frac{P(x)}{Q(x)} \) (delivered in class):

**Step 1:** Use long division to reduce the degree of the numerator until the degree of the numerator is less than that of the denominator \( Q(x) \).

**Step 2:** When the degree of \( P(x) \) is less than that of \( Q(x) \), factor \( Q(x) \) completely.

**Step 3:** Write down the partial fractions corresponding to the complete factoring of \( Q(x) \), as follows. For any factor \((ax - b)^n\), the corresponding partial fractions are

\[
\frac{A_1}{ax - b} + \frac{A_2}{(ax - b)^2} + \cdots + \frac{A_n}{(ax - b)^n}.
\]

For any factor \((ax^2 + bx + c)^n\) with \( b^2 < 4ac \), the corresponding partial fractions are

\[
\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}.
\]

**Step 4:** Use the comparison technique and/or the vanishing a factor (or variable) technique to find the constants \( A_i \)’s, \( B_i \)’s and the \( C_i \)’s.

1: (Exercise 4(b) on Page 327.) Write out the form of the partial fraction decomposition of the function \( \frac{2x + 1}{(x + 1)^3(x^2 + 4)^2} \). Do not determine the numerical value of the coefficients.

**Solution**

**Step 1:** the degree of \( 2x + 1 \) is less than the degree of \( (x + 1)^3(x^2 + 4)^2 \).

**Step 2:** The denominator has been factored completely: \( (x + 1)^3(x^2 + 4)^2 \).

**Step 3:** There will be 3 terms for \( (x + 1)^3 \) and 2 terms for \( (x^2 + 4)^2 \). The decomposition will then be

\[
\frac{2x + 1}{(x + 1)^3(x^2 + 4)^2} = \frac{A_1}{x + 1} + \frac{A_2}{(x + 1)^2} + \frac{A_3}{(x + 1)^3} + \frac{B_1x + C_1}{x^2 + 4} + \frac{B_2x + C_2}{(x^2 + 4)^2}.
\]

**Step 4:** No required.

2: (Exercise 5(a) on Page 327.) Write out the form of the partial fraction decomposition of the function \( \frac{x^4}{x^4 - 1} \). Do not determine the numerical value of the coefficients. (Hint: follow the steps taught in class).

**Solution**

**Step 1:** the degree of \( x^4 \) is the same as the degree of \( x^4 - 1 \). Long division is needed.

\[
\frac{x^4}{x^4 - 1} = \frac{x^4 - 1 + 1}{x^4 - 1} = 1 + \frac{1}{x^4 - 1}.
\]
Step 2:  Factor the denominator completely:

\[ x^4 - 1 = (x^2 + 1)(x^2 - 1) = (x^2 + 1)(x + 1)(x - 1). \]

As \( x^2 + 1 > 0 \) for any \( x \), \( x^2 + 1 \) is irreducible (cannot be further factored within real numbers).

Step 3:  There will be one term for \((x^2 + 1)\), one term for \((x + 1)\) and one term for \((x - 1)\). The decomposition will then be

\[ \frac{1}{x^4 - 1} = \frac{A_1}{x + 1} + \frac{A_2}{x - 1} + \frac{Bx + C}{x^2 + 1}, \]

or

\[ \frac{x^4}{x^4 - 1} = 1 + \frac{1}{x^4 - 1} = 1 + \frac{A_1}{x + 1} + \frac{A_2}{x - 1} + \frac{Bx + C}{x^2 + 1}. \]

Step 4:  No required.

3: (Exercise 23 on Page 327.) Write out the form of the partial fraction decomposition of the function \( \frac{10}{(x - 1)(x^2 + 9)} \). Determine all coefficients.

**Solution**

Step 1:  the degree of 10 is less than the degree of \((x - 1)(x^2 + 9)\). Long division is not needed.

Step 2:  the denominator has been factored completely: \((x - 1)(x^2 + 9)\). As \( x^2 + 9 > 0 \) for any \( x \), \( x^2 + 9 \) is irreducible (cannot be further factored within real numbers).

Step 3:  There will be one term for \((x^2 + 9)\), one term for \((x - 1)\). The decomposition will then be

\[ \frac{10}{(x - 1)(x^2 + 9)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 9}. \]

Step 4:  Combine the fraction on the right hand side above and compare the numerators both sides:

\[ 10 = A(x^2 + 9) + (Bx + C)(x - 1) \text{ or } 10 = (A + B)x^2 + (C - B)x + 9A - C. \]

**Method 1 to get \( A, B, C \): (choosing \( x \) to vanish a variable)** Let \( x = 1 \), we have \( 10 = A(1 + 9) \), and so \( A = 1 \). Let \( x = 0 \) with \( A = 1 \), we have \( 10 = 9 - C \) and so \( C = -1 \). Let \( x = -1 \) with \( A = 1 \) and \( C = -1 \), we have \( 10 = 1(1 + 9) + (-B - 1)(-2) \), and so \( B = -1 \).

**Method 2 to get \( A, B, C \): (comparing coefficients)** Comparing the coefficients to get

\[ A + B = 0, \quad C - B = 0, \quad 9A - C = 10. \]

Then \( C = B = -A \) Substitute it in \( 10 = 9A - C = 10A \), \( A = 1 \), and so \( C = B = -A = -1 \).

**Method 3 to get \( A, B, C \): (mix methods)** Let \( x = 1 \), we have \( 10 = A(1 + 9) \), and so \( A = 1 \). Let \( x = 0 \) with \( A = 1 \), we have \( 10 = 9 - C \) and so \( C = 9 - 10 = -1 \). As \( A + B = 0 \), \( B = -A = -1 \).
Any of the methods above will present

\[
\frac{10}{(x-1)(x^2+9)} = \frac{1}{x-1} - \frac{x+1}{x^2+9}.
\]

4: Evaluate the integral \( \int \frac{10}{(x-1)(x^2+9)} \, dx \).

**Solution** Use the partial fraction we obtained above.

\[
\int \frac{10}{(x-1)(x^2+9)} \, dx = \int \left( \frac{1}{x-1} - \int \frac{x+1}{x^2+9} \, dx \right) dx
\]

\[
= \int \frac{dx}{x-1} \quad \text{(Use } u = x - 1) - \int \frac{x}{x^2+9} \, dx \quad \text{(Use } w = x^2 + 9) \]

\[
- \int \frac{1}{x^2+9} \, dx \quad \text{(Use Formula 17 in the integration table)}
\]

\[
= \ln |x-1| - \frac{1}{2} \ln(x^2 + 9) - \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + C.
\]

Grade Distribution of this quiz:

**Meaning of the scores:**

- **9, 10 = Very good**, familiar with the related materials and skillful, with minimal computational errors. Keep on!
- **8 = good**, familiar with most of the related materials, with a few computations errors. Make an effort to do better.
- **7 = OK**, not so familiar with the related materials, with relatively more computational errors. We have room to improve. (For this quiz, not familiar with differentiation).
- **6 = Passing**, less familiar with the related materials and more computational errors and algebraic errors. We have lots of room to improve.
- **5 = Borderline**, We need to catch it up. If you have trouble doing your homework, it might be time to visit your instructor to get help. Do not wait to let the trouble accumulate.
- **Below 4 = This might be a dangerous warning signal. We are failing!** It should definitely be the time for us to see the instructor and get assistance to understand the materials and to practice MORE.

<table>
<thead>
<tr>
<th>Scores</th>
<th>≥ 10</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>≤ 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>5</td>
<td>6</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>Percentage</td>
<td>14.6</td>
<td>14.6</td>
<td>12.2</td>
<td>17.2</td>
<td>14.6</td>
<td>9.8</td>
<td>17</td>
</tr>
</tbody>
</table>

Discussions and Comments
(1) **An observation:** The most significant problem is that a good portion of us are NOT doing homework problems. As we may have noticed, problems in the past few quizzes are directly chosen from the homework assignment and similar exercises in the text. Those they performed poorly should take note.

**What to do:** Work on the exercises and discuss with your instructor any problems you have trouble solving. **This is very important** for us to improve our performance in the rest of course. We do not have lots of time to do that.

(2) **An observation:** I have noticed that several of us who did not do well in the past are making significant progress in this quiz, and are in the 8’s, 9’s and 10’s. It tells us that we are capable and can do excellently as long as we want to and work hard. We also observed that a number of the formerly good performers are not doing as well and falling into the lower score category. This is a warning sign.

**What to do:** For those who have made an effort to move up, keep on. You can do it! For those that have been doing well, we still need to work as before to keep the momentum on. Do not let us to be behind.

(3) **Problem:** A major common problem for those who did not do well is that we do not know what are the corresponding partial fractions for each factor in the denominator. Most in this group were in fact taking the Las Vegas approach to guess all the way. But also like going to Las Vegas, only very very few wold be lucky. This caused the major casualty in our score losing.

**What to do:** Since we know before hand that this is a quiz for partial fractions, we need to prepare for it. If you prefer to do the problems in a step by step approach, then use the steps discussed in class.

(4) **Problem:** Algebraic problems are another major score killer. Several of us knew that we need to do division for Problem 2 but we did not know how to do division. Several did not know how to solve the linear equations related in Problem 3. There were also quite a few have multiplication and addition errors.

**What to do:** With sufficient exercise and care, we can make a change and become a master of the algebra. Just practice. If you need help, come to see me.