Math 420 Homework #3
Due Wednesday, Feb. 2

1. Problems in book, Sec. 1.2, p. 31
   6a,b, 13, 14, 48

6a. \( f(x) = e^{\cos x}, f'(x) = (-\sin x)e^{\cos x}, f''(x) = (-\cos x)e^{\cos x} + \sin^2 x \cdot e^{\cos x} \)

Now we can approximate \( \ln \) using the Alternating Series Theorem, we should include the term \( \frac{e^{-\frac{\cos 1}{2}}}{2} \) and replace \( \cos - 1 \) with \( -x^2/2 + x^4/4! - ... \) multiply out and collect powers of \( x \).

6b. \( f(x) = \sin(\cos x), f'(x) = \cos(\cos x)(-\sin x), f''(x) = (-\cos x)\cos(\cos x) + (-\sin x)\sin(\cos x)(-\sin x) \)

Now we can approximate \( \ln \) using the Alternating Series Theorem, we should include the term \( (\cos 1/2)/2 \) and stop there, for the absolute error then is less than the next term, \( (\cos 1/2)/8 < \frac{1}{2} \times 10^{-8} \).

Stopping at one term previous would only guarantee an error less than \( (\cos 1/2)/7/8 \), which is not less than \( \frac{1}{2} \times 10^{-8} \).

13. \( \ln(1 + x) = x - x^2/2 + x^3/3 - ... \)

The terms are alternating and decreasing in sign, so we may apply the Alternating Series Theorem and obtain the result that the error in stopping the series at a given term is no larger than the size of the first neglected term. To guarantee an (absolute) error less than \( \frac{1}{2} \times 10^{-8} \) using the Alternating Series Theorem, we should include the term \( (\cos 1/2)/7/8 \) and stop there, for the absolute error then is less than the next term, \( (\cos 1/2)/8 < \frac{1}{2} \times 10^{-8} \).

14. Here we need the values of the function \( f(x) = x^3 - 2x^2 + 4x - 1 \) and its derivatives at \( x = 2 \). We calculate: \( f(2) = 7, f'(2) = 8, f''(2) = 6, f'''(2) = 8, f''''(2) = 6 \)

Now \( \ln(1 + x) = x - x^2/2 + x^3/3 - ... \), obtaining

\( \ln(1 + x) = \frac{1}{3}f''''(2)h^3 + \frac{1}{6}f''''(2)h^3 = 7 + 8h + 4h^2 + h^3 \)

48. You can directly calculate the Taylor series for \( f(h) = \ln(1 - (h/2)) \) as a function of \( h \), but it is more instructive to proceed from the more basic result for \( f(x) = \ln(1 + x) \):

\( \ln(1 + x) = x - x^2/2 + \frac{1}{3!}f''''(2)\xi^3 = x - x^2/2 + \frac{1}{3!}(1 + \xi)^{-3/2} x^3 \), where \( \xi \) is between 0 and x.

Now set \( x = -h/2 \), obtaining

\( \ln(1 - h/2) = -\frac{h}{2} - \frac{h^2}{8} + \frac{1}{3!}(1 + \xi)^{-3/2} \frac{h^3}{8} \), where \( \xi \) is between 0 and \( -h/2 \)

If you prefer the error in "standard form" you can set \( \xi = \eta/2 \) and obtain

\( \ln(1 - h/2) = -\frac{h}{2} - \frac{h^2}{8} + \frac{1}{3!}(1 - \eta/2)^{-3/2} \frac{h^3}{8} \), where \( \eta \) is between 0 and \( h \)

Now we can approximate \( \ln(.9998) \) by setting \( h = .0004 \) and obtain

\( \ln(.9998) = -0.0002 - \frac{(0.0004)^2}{8} = -0.0002 \times 10^{-4} \) as compared with \( \ln(.9998) = \)
- 2.000200026670447e - 004 as calculated from MATLAB.

2. a) Plot \( f(x) = \frac{1}{1 + 2e^x} \) on the interval \([-1, 1]\), along with its Taylor polynomials \( T_1 \) and \( T_2 \) about \( x = 0 \).

We will need to calculate derivatives of \( f(x) \) so here goes:

\[
f(x) = \frac{1}{1 + 2e^x}, \quad f(0) = \frac{1}{3}
\]

\[
f'(x) = -\frac{1}{(1 + 2e^x)^2} 2e^x, \quad f'(0) = -\frac{2}{9}
\]

\[
f''(x) = -2e^x \frac{1}{(1 + 2e^x)^2} + 2e^x \left[ \frac{2e^x}{(1 + 2e^x)^3} \right] = -2e^x \frac{1}{(1 + 2e^x)^2} + 8e^{2x} \frac{1}{(1 + 2e^x)^3},
\]

\[
f''(0) = -\frac{2}{9} + \frac{8}{27} = \frac{2}{27}
\]

and for later in the problem

\[
f'''(x) = -2e^x \frac{1}{(1 + 2e^x)^2} + 8e^{2x} \frac{1}{(1 + 2e^x)^3} + 16e^{2x} \frac{1}{(1 + 2e^x)^3} - 24e^{2x} \frac{2e^x}{(1 + 2e^x)^4}
\]

\[
= -\frac{2e^x}{(1 + 2e^x)^2} + \frac{24e^{2x}}{(1 + 2e^x)^3} - \frac{48e^{3x}}{(1 + 2e^x)^4},
\]

\[
f'''(0) = -\frac{2}{9} + \frac{24}{27} - \frac{48}{27} = \frac{2}{27} = .074074
\]

Now \( T_1(x) = \frac{1}{3} - \frac{2}{9}x \), \( T_2(x) = \frac{1}{3} - \frac{2}{9}x + \frac{1}{27}x^2 \)

\[
\gg f=inline('1./(1+2*exp(x))');
\]

\[
\gg x=-1:.01:1;y=f(x);
\]

\[
\gg T1=(1/3)-(2/9)*x;T2=T1+(1/27)*(x.^2);
\]

\[
\gg plot(x,y,T1,x,T2)
\]

\[
\gg legend('f(x)', 'T1', 'T2')
\]

b) On a separate plot, plot the remainder \( R_2(x) = f(x) - T_2(x) \).

\[
\gg plot(x,y-T2)
\]

\[
\gg grid on
\]
Note that the error "looks like" a multiple of $x^3$ as is predicted by Taylor’s theorem. This is discussed more precisely in the next part.

c) Taylor’s theorem predicts that in part b) $R_2(x) \approx cx^3$ for $x$ near zero. Plot \( \frac{R_2(x)}{x^3} \)
on the interval $[-1, 1]$ and show that the result is consistent with Taylor’s formula for the remainder for values of $x$ near zero and, using the mean-value form, for $x = 1$.
(Note: You may need to do a few calculations here - use your own judgment. Do not use the symbolic toolbox to calculate your derivatives.)

The remainder formula tells us that $R_2(x) = f(x) - T_2(x) = \frac{1}{3!}f'''(\xi)x^3$ where $\xi$ is between 0 and $x$. If $x$ is near zero then $\xi \approx 0$ and so $R_2(x) \approx \frac{1}{3!}f'(0)x^3$ and so

\[
\frac{R_2(x)}{x^3} \approx \frac{1}{3!}f'''(0) = \frac{1}{6} \cdot \frac{2}{27} = \frac{1}{81} = .012346
\]

The plot follows. Note the "hole" that occurs at $x=0$ because of the zero denominator. This is actually a removable discontinuity of course - the limiting value is $\frac{1}{81}$. Note that the gap and the graph around it is at the predicted value of $.012$ approximately.

>> plot(x, (y-T2)./(x.^3))

Warning: Divide by zero.

Now we need to say something about the observed value of about $.0072$ at $x = 1$ . Here we cannot be so precise and can only verify that $\frac{1}{3!}f'''(\xi) \approx .0072$. 
somewhere between 0 and $x = 1$. We simply plot $\frac{1}{3!}f'''(x)$ and look on the interval $[0, 1]$ to check this out:

```matlab
>> denom=1+2*exp(x);
>> fppp=-2*exp(x)./denom.^2+24*exp(2*x)./denom.^3-48*exp(3*x)./denom.^4;
>> plot(x,fppp/6)
>> grid on
```

![Plot of $f'''(x)$](image)

We see that $\frac{1}{3!}f'''(\xi) = 0.0072$ for some $\xi$ between 0 and 1, and in fact we can eyeball (if we zoom in) $\xi \approx 0.244$.  
