1. Calculate derivatives of the following functions:

a) \( f(x) = xe^{-x^2} \)

\[ f'(x) = e^{-x^2} + xe^{-x^2}(-2x) = (1 - 2x^2)e^{-x^2} \]

b) \( f(x) = \ln(1 + e^{-x}) \)

\[ f'(x) = \frac{1}{1 + e^{-x}} \cdot (-e^{-x}) \]

c) \( f(x) = \left(\frac{\ln x}{x}\right)^2 \)

\[ f'(x) = 2\left(\frac{\ln x}{x}\right)\left(\frac{1 + \ln x}{x^2}\right) = 2\left(\frac{\ln x}{x}\right)\left(\frac{1 + \ln x}{x^2}\right) \]

d) \( f(x) = 2^{-3x^2} = e^{-3(\ln 2)x^2} \)

\[ f'(x) = 2^{-3x^2}(-6x \ln 2) \]

e) \( f(x) = x^x = e^{x \ln x} \)

\[ f'(x) = x^x(1 + \ln x) \]
2. Evaluate the following integrals:

a) \[ \int \frac{\sin x}{1 + \cos x} \, dx = -\int \frac{1}{u} \, du = -\ln|u| = -\ln(1 + \cos x) \]

\[ u = 1 + \cos x \, , \, du = -\sin x \, dx \]

b) \[ \int 3^{-2x} \, dx = \int e^{-2(\ln 3)x} \, dx = \frac{-1}{2 \ln 3} e^{-2(\ln 3)x} = \frac{-1}{2 \ln 3} \cdot 3^{-2x} \]

c) \[ \int x \cos 2x \, dx = \frac{1}{2} x \sin 2x - \frac{1}{2} \int \sin 2x \, dx = \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x \]

\[ u = x \, ; \, du = 1 \, dx \]

\[ dv = \cos 2x \, dx \, ; \, v = \frac{1}{2} \sin 2x \]

d) \[ \int x \ln x \, dx = \frac{1}{2} x^2 \ln x - \int \frac{1}{2} x^2 \frac{1}{x} \, dx = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \]

\[ u = \ln x \, ; \, du = \frac{1}{x} \, dx \]

\[ dv = x \, dx \, ; \, v = \frac{1}{2} x^2 \, dx \]

3. a) Sketch a graph of \( f(x) = \sin^{-1} x \) ; label the important points.
Endpoints are located at \((-1,-\pi/2)\) and \((1,\pi/2)\)
b) Derive the fact that \( D(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}} \)

\[ y = \sin^{-1}x \quad \text{and we want to calculate} \quad \frac{dy}{dx}. \quad \text{We have} \quad x = \sin y \quad \text{and so} \]

\[ 1 = \cos y \frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}} \]

4. Evaluate the following limits. Indicate the indeterminate form involved at each step - derive the result, don’t just write the answer:

a) \( \lim_{x \to 0} \frac{1 - e^{-2x}}{x} = \lim_{x \to 0} \frac{2e^{-2x} - 1}{1} = 2 \)

b) \( \lim_{x \to 0} xe^{-x} = \lim_{x \to 0} \frac{x}{e^x} = \lim_{x \to 0} \frac{1}{e^x} = 0 \)

c) \( \lim_{x \to 0} x^{1/x} = \lim_{x \to 0} e^{(\ln x)/x} = e^0 = 0 \)

\[ \lim_{x \to 0} \ln x = \lim_{x \to 0} \frac{1/x}{1} = 0 \quad \text{is the limit in the exponent} \]

5) Consider the graph of \( f(x) = x \ln x \) below:
a) Locate and find the value of the absolute minimum

\[ f'(x) = \ln x + 1 = 0 \text{ when } \ln x = -1 \text{ or } x = e^{-1} \text{ and } f(x) = -e^{-1} \text{ is the value of the local minimum.} \]

b) Show that the graph is concave up for all \( x > 0 \)

Concavity is determined by the second derivative:

\[ f''(x) = 1/x > 0 \text{ for } x > 0 \text{ so graph is concave up.} \]

c) Determine/derive the behavior of \( f(x) \) as \( x \to 0 \).

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} x\ln x = \lim_{x \to 0} \frac{\ln x}{1/x} = \lim_{x \to 0} \frac{1/x}{-1/x^2} = \lim_{x \to 0} -x = 0.
\]

Extra credit: The functions \( f(x) = \frac{1}{x^{1.0000001}} \) and \( f(x) = \frac{1}{x} \) are pretty close to one another but their indefinite integrals

\[
\int \frac{1}{x^{1.0000001}} \, dx = -\frac{10000000}{x^{0.0000001}} + C \text{ and } \int \frac{1}{x} \, dx = \ln x + C \text{ look rather different.}
\]

Show how these integrals can be considered close to one another by comparing, for some fixed value of \( a \), the definite integrals \( \int_{1}^{a} \frac{1}{x^{1+\epsilon}} \, dx \) and \( \int_{1}^{a} \frac{1}{x} \, dx \) as \( \epsilon \to 0 \).

\[
\int_{1}^{a} \frac{1}{x^{1+\epsilon}} \, dx = \frac{1}{\epsilon} \left( 1 - a^{-\epsilon} \right) \text{ as compared with } \int_{1}^{a} \frac{1}{x} \, dx = \ln a.
\]

Now

\[
\lim_{\epsilon \to 0} \frac{1 - a^{-\epsilon}}{\epsilon} = \lim_{\epsilon \to 0} \frac{(\ln a)a^{-\epsilon}}{1} = \ln a \text{ so the two integrals are close to each other when } \epsilon \text{ is small.}
\]

In the example above, if we write

\[
\int \frac{1}{x^{1.0000001}} \, dx = -\frac{10000000}{x^{0.0000001}} + 1000000 \text{ and } \int \frac{1}{x} \, dx = \ln x \text{ then these}.
\]
two antiderivatives are in fact extremely close to each other unless $x$ is very close to zero or very large. If we consider the much less extreme case of \[ \int \frac{1}{x^{1.01}} \, dx = \frac{-100}{x^{0.01}} + 100 \] as compared with \[ \int \frac{1}{x} \, dx = \ln x, \] we see the two graphs below compare very well.