Review topics for Exam 4

Sequences, series, and power series

Sequences, convergent sequences, formulas for our favorite sequences (even numbers, odd numbers, powers, factorials, alternating sign)

Infinite Series:

Summation (sigma) notation. Writing out the first few terms.
What does it mean when we say an infinite series converges?
Manipulations with series: Sum of two series, splitting a series up, multiplying by constants

Basic fact about \( \sum_{n=1}^{\infty} a_n \): If \( \lim_{n \to \infty} a_n = 0 \) is not true then the series does not converge.

Important special series:

Geometric series: \( \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots = \frac{a}{1-r} \) if the ratio \( r \) satisfies \( |r| < 1 \). Various forms in which geometric series can be written.

\( p \)-series: \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges when \( p > 1 \), diverges otherwise

Tests for analyzing whether or not a series converges:
Comparison test:
If a series of "bigger terms" converges then original series converges (for general series)
If a series of "smaller terms" diverges then original series diverges (for positive series)
Any comparison which is "eventually" true (that is, true for large enough values of \( n \) ) can be used.

Limit comparison test (for positive series): If \( \lim_{n \to \infty} \frac{a_n}{b_n} = L \) and \( 0 < L < \infty \) then \( \Sigma a_n \) and \( \Sigma b_n \) are "doing the same thing" (both diverge or both converge) If \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \) and \( \Sigma b_n \) converges then \( \Sigma a_n \) converges. If \( \lim_{n \to \infty} \frac{a_n}{b_n} = \infty \) and \( \Sigma b_n \) diverges then \( \Sigma a_n \) diverges.
Important limits for classes of functions: The further down the function the faster it grows.
1) $(\ln n)^a$, $a > 0$ (for example $(\ln n)^2, \sqrt{\ln n}$)
2) $n^\beta$, $\beta > 0$ (for example $n^3, \sqrt{n}$)
3) $a^{n^\gamma}$, $a > 1$, $0 < \gamma \leq 1$ (for example $2^n, e^n, 2\sqrt{n}$)
4) $n!$
5) $n^n = e^{n\ln n}$
6) $a^{n^\delta}$, $a > 1$, $\delta > 1$ (for example $2^{n^2}$)

Alternating series test: Series that alternate in sign, with terms that decrease in size to zero are convergent series.

Absolute convergence: If $\Sigma |a_n|$ converges then $\Sigma a_n$ converges and we say that $\Sigma a_n$ converges absolutely. This is useful for series of positive and negative terms - just look at the absolute value of the terms and if the series of absolute values converges then the original series converges. (But if the series of absolute values diverges you can’t necessarily conclude that $\Sigma a_n$ diverges)

Ratio test: If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = r$ and $r < 1$ then $\Sigma a_n$ converges absolutely, while if $r > 1$ the series diverges. Useful for series involving terms like $a^n$ and $n!$

Root test: If $\lim_{n \to \infty} |a_n|^{1/n} = r$ and $r < 1$ then $\Sigma a_n$ converges absolutely, while if $r > 1$ the series diverges. Useful for series involving terms like $(junk)^n$

Power series:
- Finding the radius of convergence.
- Manipulations/operations/tricks/fun with power series.
- Differentiating and integrating power series.
- Taylor series: A power series in powers of $(x - a)$ associated with a function $f(x)$ and its derivatives at $x = a$. Taylor series of $\sin x, \cos x, e^x, \ln(1 + x)$