Math 155 Worksheet 8 Solutions

1. Linear approximation of \( f(x) = \frac{1}{x^2} \) as \( x_0 = 1 \). Graph function and linear approximation.

\[ L(x) = f(x_0) + f'(x_0)(x-x_0) \]
\[ f(x_0) = 1, \quad f'(x) = -\frac{2}{x^2}, \quad f'(x_0) = -2 \]
\[ L(x) = 1 - 2(x - 1) \]

The graph of \( L(x) \) is tangent to graph of \( f(x) \) at \( x = x_0 \)

2. Estimate \((80)^{\frac{1}{4}}\).

Note that \((81)^{\frac{1}{4}} = 3\) so consider \( f(x) = x^{\frac{1}{4}} \) near \( x = x_0 = 81 \).

The linear approximation is obtained from:

\[ f(x_0) = 81^{\frac{1}{4}} = 3, \quad f'(x) = \frac{1}{4}x^{-\frac{3}{4}}, \quad f'(x_0) = \frac{1}{4}(81)^{-\frac{3}{4}} = \frac{1}{108} \]

and \( x^{\frac{1}{4}} \approx L(x) = 3 + \frac{1}{108}(x - 81) \) and now plug in \( x = 80 \) to obtain

\((80)^{\frac{1}{4}} \approx 3 - \frac{1}{108}\)

3. Apply L’Hopital’s rule or show that limit doesn’t exist

a) \[ \lim_{x \to 0} x^3 - 1 \]

\[ \lim_{x \to 0} \frac{e^{3x} - 1}{x} = \lim_{x \to 0} \frac{3e^{3x}}{1} = 3 \]

b) \[ \lim_{x \to 0} \frac{\tan x - \sin x}{x} \]

\[ \lim_{x \to 0} \frac{\sec^2 x - \cos x}{x} = \lim_{x \to 0} \frac{2\sec^2 x \tan x - \sin x}{2} = 0 \]

4. Find all critical numbers

a) \( f(x) = x^2e^x \) domain is all \( x \) and is differentiable for all \( x \)

\[ f'(x) = 2xe^x + x^2e^x = e^x(x + 2) = 0, \quad x = 0, -2 \] are critical numbers

b) \( f(x) = \frac{x^2 - x + 4}{x - 1} \)

\[ f'(x) = \frac{(x - 1)(2x - 1) - (x^2 - x + 4)}{(x - 1)^2} = \frac{x^2 - 2x - 3}{(x - 1)^2} \]

\( (x - 3)(x + 1) = 0, \quad x = -1, 3 \) are critical points

5. Find absolute extrema
a) \( f(x) = 2x^3 + 3x^2 + 4 \), \([-2, 1]\)

\[
 f'(x) = 6x^2 + 6x = 6x(x + 1)
\]
critical points \( x = 0, -1 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
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<tbody>
<tr>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>5</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
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Absolute min is \( f(-2) = 0 \); absolute max is \( f(1) = 9 \)

b) \( f(x) = x^{4/5} \) on \([-1, 32]\)

\[
 f'(x) = \frac{4}{5}x^{-1/5}
\]
critical points \( x = 0 \) (deriv doesn’t exist)

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<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
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<tbody>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>32</td>
<td>16</td>
</tr>
</tbody>
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Absolute min \( f(0) = 0 \); absolute max \( f(32) = 16 \)

6. \( f(x) = x^4 - 8x^2 \)

a) Critical numbers: \( f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x + 2)(x - 2) = 0 \)
\( x = 0, 2, -2 \) are critical numbers

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f'(x) )</th>
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<tbody>
<tr>
<td>-2</td>
<td>-</td>
</tr>
<tr>
<td>0</td>
<td>+</td>
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<td>2</td>
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\( f \) increasing on \((-2, 0)\), \((2, \infty)\)

\( f \) decreasing on \((-\infty, -2)\), \((0, 2)\)

c) \( f(-2) = -16 \) is local min
\( f(0) = 0 \) is local max
\( f(2) = -16 \) is local min