Worksheet 3 Solutions

1. Sketch the graph of a function $f$ with the given properties:
   the domain of $f$ is $[0,4]$, $\lim_{x \to 1} f(x) = 1$, $\lim_{x \to 1} f(x) = 3$, and $\lim_{x \to 2} f(x) = 2$.

2. In each of the following, find the limit or show that the limit does not exist:
   (a) $\lim_{x \to 2} \sqrt{4x+1}$
   (b) $\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3}$
   (c) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$ (problem 24, page 100)
   a) $\lim_{x \to 2} \sqrt{4x+1} = \sqrt{9} = 3$
   b) $\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(x - 2)}{x - 3} = \lim_{x \to 3} (x - 2) = 1$
   c) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \lim_{x \to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$
3. Let \( f(x) = \begin{cases} 
2x + 1 & \text{if } x < 1 \\
-3x + 5 & \text{if } x \geq 1 
\end{cases} \). Find each of the following limits, or explain why the limit does not exist.

(a) \( \lim_{x \to 1} f(x) \)
(b) \( \lim_{x \to 1^+} f(x) \)
(c) \( \lim_{x \to 1^-} f(x) \)

\( a) \lim_{x \to 1} f(x) = 3 \quad b) \lim_{x \to 1} f(x) = 2 \)
\( c) \lim_{x \to 1} f(x) \) does not exist because \( \lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x) \)

4. (a modified version of problem 33, page 100) Use the Squeeze Theorem to prove that
\[ \lim_{x \to 0} x^2 \sin \left( \frac{1}{x} \right) = 0. \]

\[ \left| x^2 \sin \left( \frac{1}{x} \right) \right| \leq x^2 \text{ and } \lim_{x \to 0} x^2 = 0 \text{ so } \lim_{x \to 0} x^2 \sin \left( \frac{1}{x} \right) = 0 \text{ by the Squeeze Theorem} \]

5. Let \( f(x) = \begin{cases} 
5x - c & \text{if } x \leq 1 \\
3cx^2 + 1 & \text{if } x > 1 
\end{cases} \). Find the number \( c \) which makes the function \( f \) continuous.

\( \lim_{x \to 1^-} f(x) = 5 - c \), \( \lim_{x \to 1^+} f(x) = 3c + 1 \) so \( \lim_{x \to 1} f(x) \) exists if \( 5 - c = 3c + 1 \), \( c = 1 \) in which case \( \lim_{x \to 1} f(x) = 5 - c = 4 = f(1) \) so \( f \) is continuous at 1 and at all other values of \( x \) as well, and so \( f \) is continuous.

6. Let \( f(x) = \frac{x^2 + 3x + 2}{x^2 - 4} \). Find all discontinuities of the function \( f \). Classify each discontinuity as removable or nonremovable. For each discontinuity that is removable, define a new function which removes the discontinuity.

For \( f(x) = \frac{x^2 + 3x + 2}{x^2 - 4} \) the discontinuities are located where the denominator is zero, namely \( x = \pm 2 \). Now \( \frac{x^2 + 3x + 2}{x^2 - 4} = \frac{(x+1)(x+2)}{(x-2)(x+2)} = \frac{x+1}{x+2} \) for \( x \neq \pm 2 \). So \( \lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x+1}{x+2} = \frac{3}{4} \) while \( \lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{x+1}{x+2} \) which does not exist. So \( x = 2 \) is a removable discontinuity, while \( x = -2 \) is not removable and the function \( \frac{x+1}{x+2} \) defines a new function which is continuous at \( x = 2 \) and is equal to \( f(x) \) on that function's original domain.