Math 155       Worksheet 1

1. Solve each of the following inequalities:
   (a) \( x^2 - 3x - 10 < 0 \)

   (b) \( |2x - 3| > 5 \)

   (c) \( \frac{x - 2}{(x + 2)^2} > 0 \) (problem 32, page 9)

In general, to solve an inequality involving \( f(x) > 0 \) or similar problems \( \langle 0, \leq 0, \geq 0 \) we find the points where \( f(x) = 0 \) or is not defined. For the functions we normally deal with, \( f(x) \) will then have one sign in between adjacent points which should be easy to determine. Also in such problems, it helps to factor \( f(x) \) in so far as possible.

a. \( f(x) = x^2 - 3x - 10 = (x - 5)(x + 2) \)
   \( x^2 - 3x - 10 = 0 \), \( (x - 5)(x + 2) = 0 \), \( x = -2, 5 \)

   Sign chart of \( f(x) \) is shown below:
   
   \[
   \begin{array}{cccccccccccc}
   f(x) & + & + & 0 & - & - & 0 & + & + \\
   x & & & & & & & & & & \\
   & & -2 & & & & & & & 5 & \\
   \end{array}
   \]

   Explanation: Examining \( f(x) = (x - 5)(x + 2) \), for \( x > 5 \) both factors are positive so \( f(x) \) is positive. As we pass over \( x = 5 \), the factor \( (x - 5) \) changes sign, so that \( f(x) \) changes sign to become negative. As we pass over \( x = -2 \) the factor \( (x + 2) \) changes sign, so \( f(x) \) changes sign from negative to positive. As a check, you can also evaluate \( f(x) \) at any point in an interval to determine its sign in that interval.

   Answer: \( f(x) < 0 \) for \( x > -2 \) and \( x < 5 \), which we can write as \(-2 < x < 5 \)

   b. \( |2x - 3| > 5 \)

   Two approaches:
   
   First we make use of what we know about absolute value, i.e. \(|u| > a \) if and only if \( u > a \) or \( u < -a \). In this case \(|2x - 3| > 5 \) if and only if \( 2x - 3 > 5 \) or \( 2x - 3 < -5 \), resulting in \( x > 4 \) or \( x < -1 \), which we can also write as \((-\infty, -1) \cup (4, \infty) \)

   We can also use the approach of part a: Set \( f(x) = |2x - 3| - 5 \). \(|2x - 3| - 5 = 0 \) has solution \( 2x - 3 = 5 \), \( x = 4 \) or \( 2x - 3 = -5 \), \( x = -1 \). It’s then easy to verify that \(|2x - 3| - 5 \) has the following sign chart:

   \[
   \begin{array}{cccccccccccc}
   f(x) & + & + & 0 & - & - & 0 & + & + \\
   x & & & & & & & & & & \\
   & & -1 & & & & & & & 4 & \\
   \end{array}
   \]

   and we obtain the same result using this approach.
c. $\frac{x-2}{(x+2)^3} > 0$

Let $f(x) = \frac{x-2}{(x+2)^3}$. The important points, where $f(x) = 0$ or is undefined, are at $x = 2$ (where $f(x) = 0$) and $x = -2$ (where $f(x)$ is undefined). We can then draw the picture:

$$f(x) + + * - - - 0 + +$$

Note that all factors in $f(x)$ are positive for $x > 2$, then the factor $(x-2)$ changes sign as we move to the left, crossing $x = 2$, and then the factor $(x+2)^3$ in the denominator changes sign as we cross $x = -2$. So the answer is $\frac{x-2}{(x+2)^3} > 0$ for $(-\infty,-2) \cup (2,\infty)$

2. Find an equation of the line through the points $(-2,3)$ and $(4,-5)$. Write your answer in the form $y = mx + b$.

We first find the slope: $m = \frac{(5) - 3}{4 - (-2)} = -\frac{4}{3}$. Then the equation is

$$y = 3 + \left(-\frac{4}{3}\right)(x - (-2)) = \frac{1}{3} - \frac{4}{3}x$$

(you might want to check your answer as well)

3. Find an equation of the line through $(5,2)$ which is perpendicular to the line $3x - y = 6$.

Write your answer in the form $y = mx + b$.

First, write the original line in the form $y = mx + b$ to determine its slope $m$:

$y = 3x - 6$ so the original slope is 3 and the perpendicular line has slope $-\frac{1}{3}$. The equation of the perpendicular line, using the point $(5,2)$, is then

$$y = 2 + \left(-\frac{1}{3}\right)(x - 5) = \frac{1}{3} - \frac{1}{3}x$$

4. Find the domain of each of the following functions:

(a) $f(x) = \frac{1}{\sqrt{x-4}}$

(b) $f(x) = \frac{x}{x^2-x-6}$

a) You can't have zeros in the denominator of a fraction, and you can't take the square root (or more generally, an even root) of a negative number. So in a) $x > 4$ is the domain.

b) We just need to avoid the points where $x^2 - x - 6 = 0$, $(x-3)(x+2) = 0$, $x = 3, -2$ so the domain is $\{x | x \neq -2, 3\}$
5. Solve each of the following equations:

(a) $6x^2 + x - 15 = 0$

(b) $x^2 - 6x + 3 = 0$

(c) $x^4 - 7x^2 + 12 = 0$

a) $6x^2 + x - 15 = 0$
You can try to factor the left side, but it may not generally factor in integers and the quadratic formula always works, so if I can’t see a factorization in a few seconds, I go to the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 + 360}}{12} = \frac{-1 \pm 19}{12} = \frac{3}{2}, \quad -\frac{5}{3}$$

There is also the factorization $(2x - 3)(3x + 5)$ now that we see that the roots are rational and we know their value.

b) $x^2 - 6x + 3 = 0$ Here there is no factorization using integers. The quadratic formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{36}}{2} = -3 \pm \sqrt{6}$$

c) $x^4 - 7x^2 + 12 = 0$. Here the trick is to think of $x^2$ as a variable $u$, so that $x^4 = u^2$ and $x^4 - 7x^2 + 12 = u^2 - 7u + 12 = (u - 4)(u - 3) = 0$, $u = 3, 4$ so

$x^2 = 3, 4$ and $x = \pm \sqrt{3}, \pm 2$

(You can also directly factor $x^4 - 7x^2 + 12 = (x^2 - 4)(x^2 - 3)$ and proceed to the same solution.)