First we draw the radii OS, OT, and observe that OS = OT and OS ⊥ PS, OT ⊥ PT from tangency. Therefore, ΔOSP and ΔOTP are right triangles with \( \angle OSP = \angle OTP = \frac{\pi}{2} \). Next, we also know that OP = OP.

So we can deduce that SP = TP from Pythagorean Theorem. Finally, ΔOSP ≅ ΔOTP from SSS and \( \angle PSD = \angle TPD \), making OP the angle bisector of \( \angle SPT \).

We have two circles meeting at points P and Q and diameters PA, PB. We know \( \angle PAB \) is half the arc it subtends so \( \angle PAB = \frac{\pi}{2} \) since PB is the diameter. Similarly, \( \angle PBA = \frac{\pi}{2} \).

Since \( \angle ABQ = \angle AQP + \angle PAB = \frac{\pi}{2} + \frac{\pi}{2} = \pi \), A, Q, B are collinear and AB goes through point Q.

Let U, V be points where bisector OY meets secant PD and PC, respectively. In inscribed quadrilateral ABCD, we label \( \angle B = \alpha \) and \( \angle C = \beta \), and deduce \( \angle A = \pi - \alpha \) and \( \angle B = \pi - \beta \) since opposite angles are supplementary. In ΔBCX, we compute \( \angle BPC = \pi - \alpha - \beta \). In ΔABQ, we compute \( \angle AOD = \pi - \alpha - (\pi - \beta) - \beta - \alpha \). In ΔVBC, we know \( \angle YQC = \pi - \beta \) from the straight angle, \( \angle VQC = \frac{1}{2}(\beta - \alpha) \) from the angle bisector, so \( \angle VQC = \pi - (\pi - \beta) - \frac{1}{2}(\beta - \alpha) = \frac{1}{2}(\beta + \alpha) \).

Since \( \angle PUV \) and \( \angle QVC \) are vertical angles, \( \angle PUV = \frac{1}{2}(\beta + \alpha) \). Now look at ΔPUV, we can compute \( \angle PUV = \pi - (\pi - \alpha - \beta) - \frac{1}{2}(\beta + \alpha) = \frac{1}{2}(\beta + \alpha) \).

Hence \( \angle PUV = \angle QVC = \frac{1}{2}(\beta + \alpha) \). From the converse of Pons Asinorum, PU ≠ PV, and ΔPUV is isosceles. Furthermore, we know angle bisector is also the altitude in an isosceles triangle, so PD ⊥ QY (the two sub-triangles are congruent from ASA so the two adjacent angles summing to the straight angle are congruent right angles).
Solutions to Problem Set 4, Math 461, Spring 2010

Problem 1 (1F.3) Let $P$ be a point exterior to a circle centered at point $O$ and draw the two tangents to the circle from $P$. Let $S$ and $T$ be the two points of tangency. Show that $OP$ bisects $\angle SPT$ and $PS = PT$.

Solution Since $PS$ is tangent to the circle at $S$, we know that $PS$ is perpendicular to $OS$. Similarly, $PT$ is perpendicular to $OT$. By the hypotenuse-arm congruence theorem, we get $\triangle OPT \cong \triangle OPS$. Then, by corresponding parts, we find $PS = PT$ and $\angle SPO \cong \angle TPO$, so that $OP$ is a bisector of $\angle SPT$.

Problem 2 (1F.8) In the figure at right, two circles meet at points $P$ and $Q$, and diameters $PA$ and $PB$ are drawn. Show that $AB$ goes through point $Q$.

Solution We need to show that $\angle PQA$ and $\angle PQB$ are supplementary angles. Since $PA$ is a diameter, we get that $\angle PQA$ is a right triangle. Similarly, we conclude $\angle PQB$ is a right angle. Since $\angle PQA$ and $\angle PQB$ are both right angles, they are supplementary. Thus $Q$ lies on $AB$.

Problem 3 (1F.12) In the figure at left, we have selected two points $P$ and $Q$ outside of a circle, and we have drawn two secants through each point in such a way that these four secants intersect the circle in the four points $A$, $B$, $C$, and $D$, as shown. Show that the angle bisectors $PX$ and $QY$ of $\angle P$ and $\angle Q$ are perpendicular to each other.

Solution Let $Y''$ and $Y'$ be points on the line $QY$ that intersect the circle, with $Y'$ between $Q$ and $Y''$. We then see that

$$\angle PUV = \frac{1}{2}(\widehat{DY''} + \widehat{AY'}) = \frac{1}{2}(\widehat{DY''} + \widehat{AB} + \widehat{BY'})$$

and

$$\angle PVU = \frac{1}{2}(\widehat{CY'} + \widehat{BY''}) = \frac{1}{2}(\widehat{CY'} + \widehat{AB} + \widehat{AY''}).$$

Then we subtract:

$$\angle PUV - \angle PVU = \frac{1}{2}(\widehat{DY''} + \widehat{AB} + \widehat{BY'}) - \frac{1}{2}(\widehat{CY'} + \widehat{AB} + \widehat{AY''})$$

$$= \frac{1}{2}(\widehat{DY''} - \widehat{CY'}) - \frac{1}{2}(\widehat{AY''} - \widehat{BY'})$$

$$= \angle DQU - \angle AQU$$

$$= 0,$$

the last equality following from the hypothesis that $QY$ bisects $\angle Q$. Thus $\angle PUV \cong \angle PVU$, so $\triangle PUV$ is isosceles by the converse to pons asinorum. The bisector of an isosceles triangle is perpendicular to the base, so $PX$ is perpendicular to $QY$. 