Exponential Functions: Data Analysis

- Section 4.1

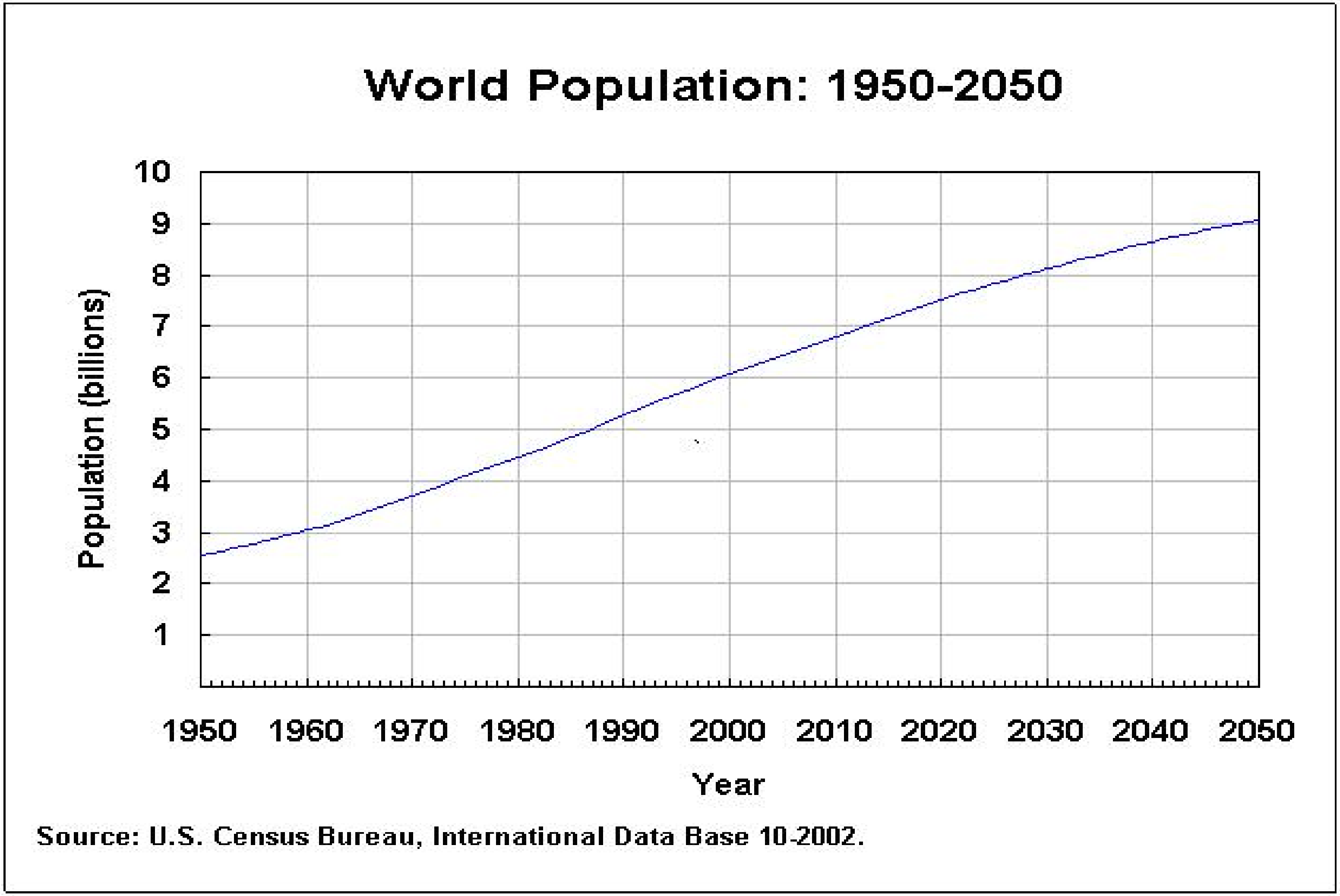
- Transcendental Functions
  - Transcendental comes from the Latin *trans* meaning “across, beyond” and *scrandere* meaning “to climb”.
  - Transcendental functions climb beyond algebraic functions – they are non-algebraic.

Model: World Population

- The U.S. Census Bureau provides estimates of the world population in billions (http://www.census.gov People International World Pop.)
  - Find a model for world population with respect to time.
  - Estimate the World Population in 2005.

Year	Pop.
1950	2.556
1955	2.780
1960	3.039
1965	3.346
1970	3.708
1975	4.087
1980	4.455
1985	4.850
1990	5.275
1995	5.685
2000	6.079

World Population Graph



Model: World Population

- Try fitting an algebraic model
  - $P(t) = 0.0004\, t^2 + 0.05\, t + 2.51$
  - Standard Deviation is 0.029
- Despite the good fit, we will use a well established biological law to model the data
  - Supports intuitive idea that growth should be related to the initial population.

Biological Law for Population Growth

$$r_i = k \cdot p_i$$

- The rate of growth r_i of a population over any time interval i is proportional to the population p_i at the beginning of that time interval.
  - Growth: $g_i = p_{i+1} - p_i$
  - Rate of Growth: $r_i = \frac{g_i}{\Delta t}$
  - Rate of Increase: $k_i = \frac{r_i}{p_i}$

Constant Rate of Increase

- The Biological Law of Population Growth implies that the rate of increase k will be constant
    - Calculate the rate of increase for the world population data. Is it relatively constant? If not, why not?

Calculation - k for first interval

- For the first interval change in time is

  $\Delta t = 5 - 0 = 5$, $p_1 = 2.556$ and $p_2 = 2.78$
- Growth: $g_1 = p_2 - p_1 = 2.78 - 2.556 = 0.224$
- Rate of Growth:

$$r_1 = \frac{g_1}{\Delta t} = \frac{0.224}{5} = 0.0448$$

- Rate of Increase:

$$k_1 = \frac{r_1}{p_1} = \frac{0.0448}{2.556} \approx 0.018$$

Year	**Pop.** p	Growth g	Rate of Growth r	Rate of Increase k
1950	2.556			
1955	2.780	g_1=0.224	r_1=0.045	k_1=0.018
1960	3.039	g_2=0.259	r_2=0.052	k_2=0.019
1965	3.346	g_3=0.307	r_3=0.061	k_3=0.020
1970	3.708	g_4=0.362	r_4=0.072	k_4=0.022
1975	4.087	g_5=0.379	r_5=0.076	k_5=0.020
1980	4.455	g_6=0.368	r_6=0.074	k_6=0.018
1985	4.850	g_7=0.395	r_7=0.079	k_7=0.018
1990	5.275	g_8=0.425	r_8=0.085	k_8=0.018
1995	5.685	g_9=0.410	r_9=0.082	k_9=0.016
2000	6.079	g_{10}=0.394	r_{10}=0.079	k_{10}=0.014

Rate of Increase k

- Since k is relatively constant let's average the values of k to get a constant k = 0.018
  - Difficult to determine algebraic model
  - Can use the constant to find the population for the next time interval given a time interval

Recurrence Function

- Let $p(0)$ = initial population
- Growth per year for this interval is r_1
- Then $p(1)$ = original pop. + growth

$$= p(0) + r_1 \cdot \Delta t$$

$$= p(0) + k \cdot p(0) \cdot \Delta t$$

$$= p(0) \cdot (1 + k \cdot \Delta t)$$

Recurrence Function

- A function defining the nth term in terms of previous terms
- General recurrence function for World Population
- $p(n) = p(n-1) + r_n \cdot \Delta t$

$$= p(n-1) + k \cdot p(n-1) \cdot \Delta t$$

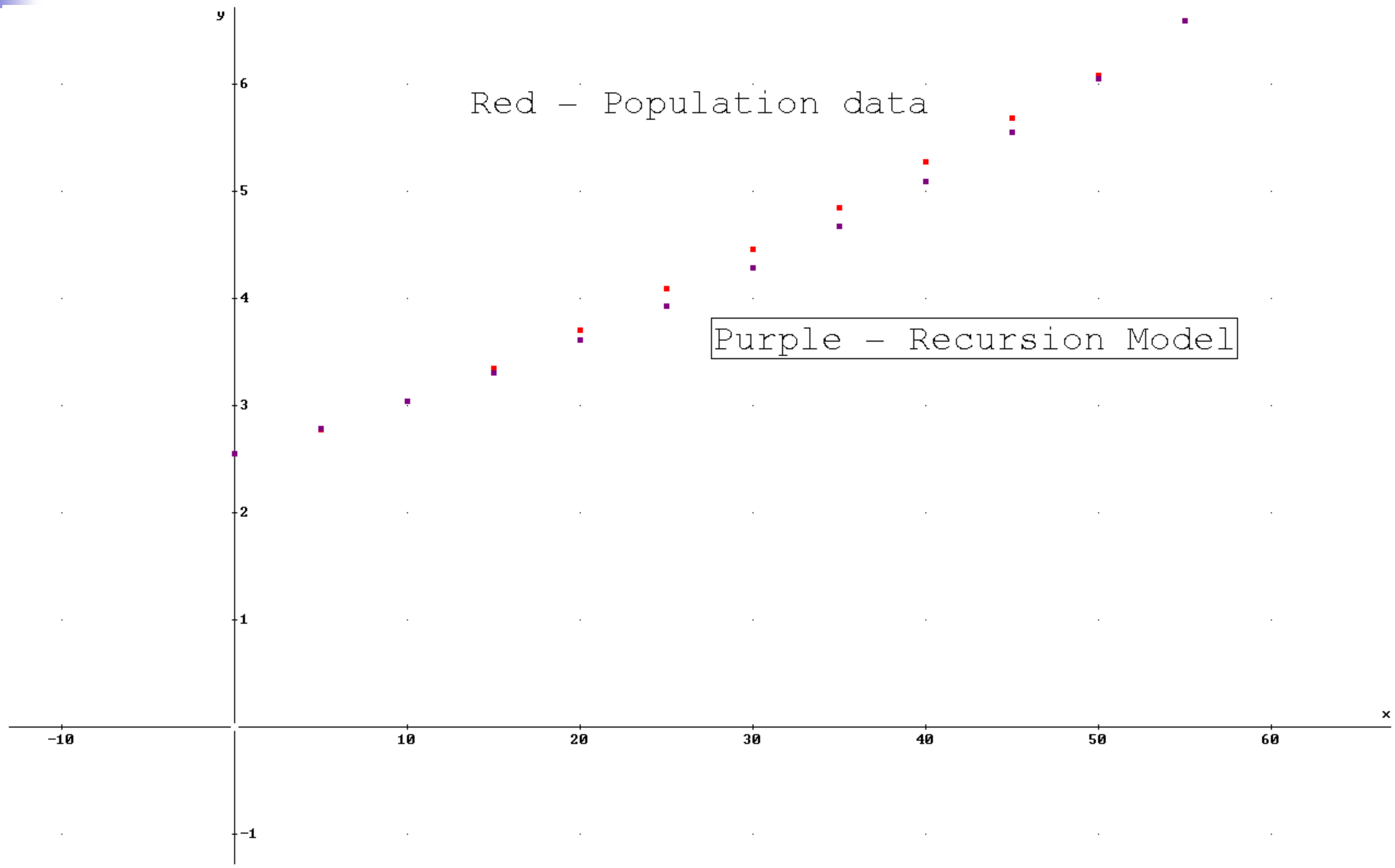
$$= p(n-1) \cdot (1 + k \cdot \Delta t)$$

Recurrence Function Drawback

- If we use the recurrence function for World Population to estimate the population in the year 2005, what values must we find first?
- Use technology to implement the recurrence function and find the predicted population in the year 2005.

- Why could implementing recurrence functions be a problem?

Recursive Model (2005, 6.596)



Closed Form Model

- Closed form of a recurrence function is one that is written in terms of only the independent variable and known constants.

  - Find the closed form for the World Population model
  - Solution: $p(n) = p(0)\,(1 + k \cdot \Delta t\,)^n$

    $p(n) = 2.556\,(1 + 0.018 \cdot 5\,)^n$

    $p(n) = 2.556\,(\,1.09\,)^n$

Exponential Function Characteristics

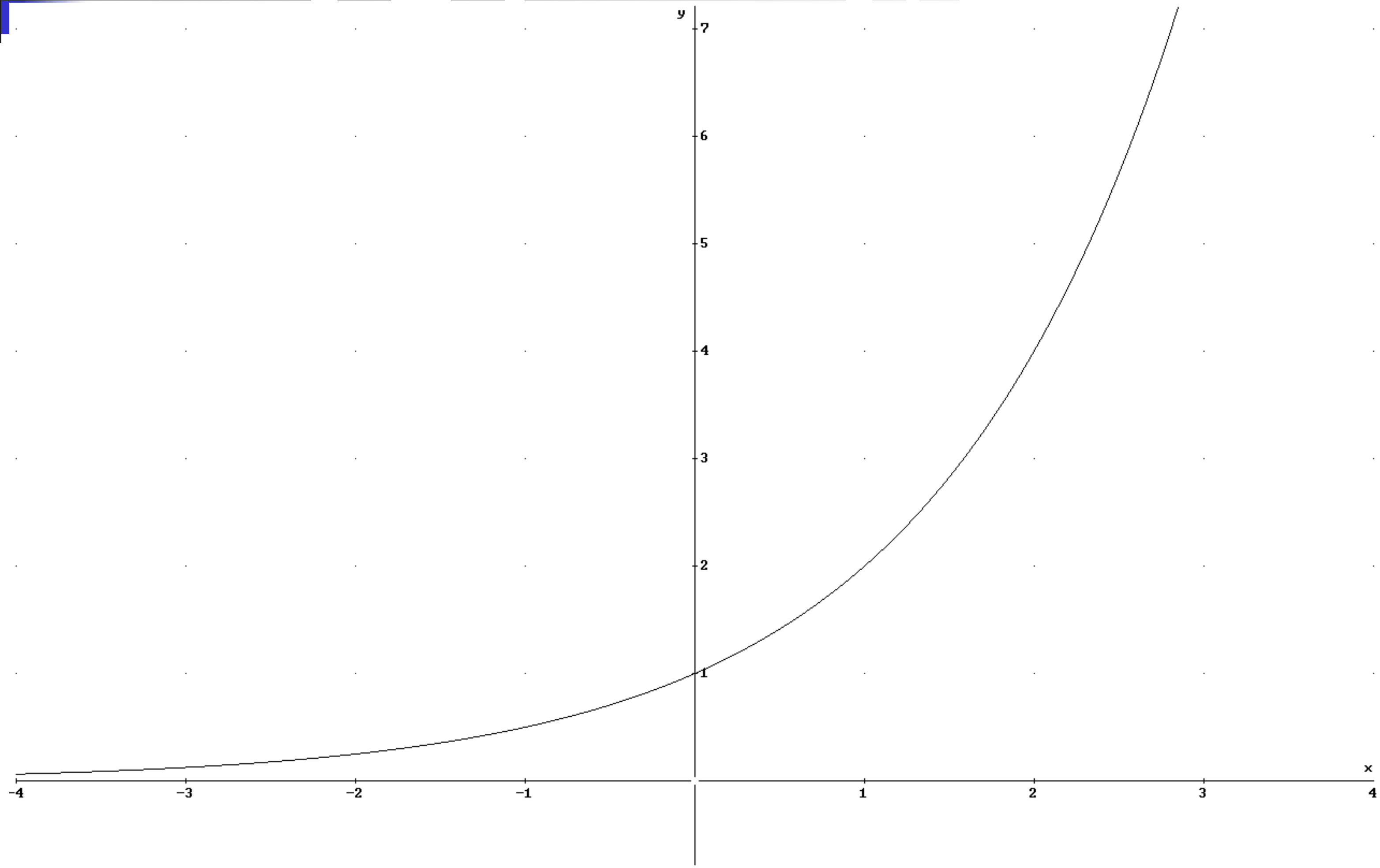
$$f(x) = a \cdot b^x$$

where the base *b* is a positive real number other than 1.

- What is the end behavior of the exponential function?
- What is the intermediate behavior of the exponential function?
- Why restrict the exponential function from having a negative base *b*? Why not let *b* equal 0 or 1?
- Which grows faster, an exponential function or a power function?

Exponential Function Characteristics

$$f(x) = 2^x$$



Exponential Function

- How can we determine if data is exponential?
  - Find k and determine if it is constant.
  - Determine if the ratio of consecutive values is constant. The constant will be the base.

$$\frac{a \cdot b^{x+1}}{a \cdot b^{x}} = b$$

Exponential Function

- Participation Exercise: Determine if the given data is exponential?
  - Find k and determine if constant.
  - SOLUTION: k = 2 so
    $$y = 2 \cdot (1+2)^x = 2 \cdot 3^x$$
  - Determine if the ratio of consecutive values is constant.
  - SOLUTION: ratio is b = 3 so
    $$y = a \cdot 3^x$$
    $$2 = a \cdot 3^0 \; so \; a = 2$$

x	y
0	2
1	6
2	18
3	54
4	162
5	486
6	1458
7	4374
8	$1.3122 \cdot 10^4$
9	$3.9366 \cdot 10^4$
10	$1.18098 \cdot 10^5$