Mathematical Modeling – Finite Differences

- Section 2.2

  - Charles Babbage (England, 1821) created a forerunner of the computer called the Difference Engine.
  - Based his discovery on looking for constant value by taking differences
  - Basic premise of math is to determine what remains constant within change

Model: Guess My Rule

- An old mathematical game where one person makes up a rule and generates data, then a second person tries to guess the rule.
  - How can we determine if the rule is linear or curvilinear?
  - Solution: Determine if the rate of change is constant.

n	F(n)
0	7
1	10
2	13
3	16
4	19
5	22
6	25
7	28
8	31
9	34
10	37
11	40
12	43
13	46
14	49
15	52
16	55
17	58
18	61
19	64
20	67

Finite Differences

- Find differences between successive terms in a sequence of numbers until a common difference occurs.
- If the data is modeled by a polynomial function (linear, quadratic, or cubic, etc.), then there will be a common difference.
    - Solution: Common difference in 1^{st} difference, so model is linear

n	F(n)	1^{st} Difference
0	7	
		10 – 7 = 3
1	10	
		13 – 10 = 3
2	13	
		16 – 13 = 3
3	16	
		19 – 13 = 3
4	19	
		22 – 19 = 3
5	22	
		25 – 22 = 3
6	25	
		28 – 25 = 3
7	28	
		31 – 28 = 3
8	31	
		34 – 31 = 3
9	34	
		37 – 34 = 3
10	37	

Finite Differences Model

- Compare the table of differences for the data to the general finite differences table for the linear case $f(x) = mx + b$ (Table 4, Pg 260)
  - Generate the Linear Case table by letting x assume values 0, 1, 2, 3, 4, 5, …..
  - Since the data is linear and the Linear Case table represents the general pattern for any line, the entries in the table must be equal.
  - Select a line of the table, set the entries equal, and solve for m and b.
  - Solution: My rule was $F(n) = 3n + 7$

When is finite differences a good method to use?

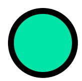
- Theoretical data with no scatter due to variation or measurement error
  - Example: mathematical sequences
- Scientific data with little scatter due to measurement error
  - Distance an object falls in a given time
- NOT GOOD for Social Science data which often has a lot of variation
  - Example: Income level by age

Curvilinear Case

- Given n points in a plane, what is the maximum number of straight line segments (edges) that can be drawn joining them?
  - Gather data for n = 1, 2, 3, 4 and 5 points
  - Is the data linear? Why or why not?
  - If the data is curvilinear, should we use a quadratic or cubic polynomial to model it?

Data for edges problem

n = 1 point

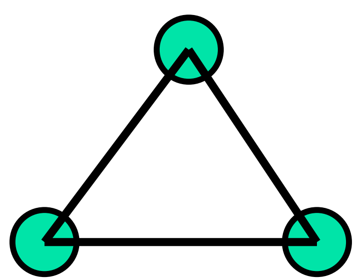


e = 0 edges

n = 2 points

e = 1 edge

n = 3 points



e = 3 edges

Find the number of edges for n = 4 and n = 5.

Data for Edges Problem

- Here is the data for the first n = 8 cases of the edges problem.
  - Use finite differences to determine if the data is linear or curvilinear.
  - Solution: Data is curvilinear.

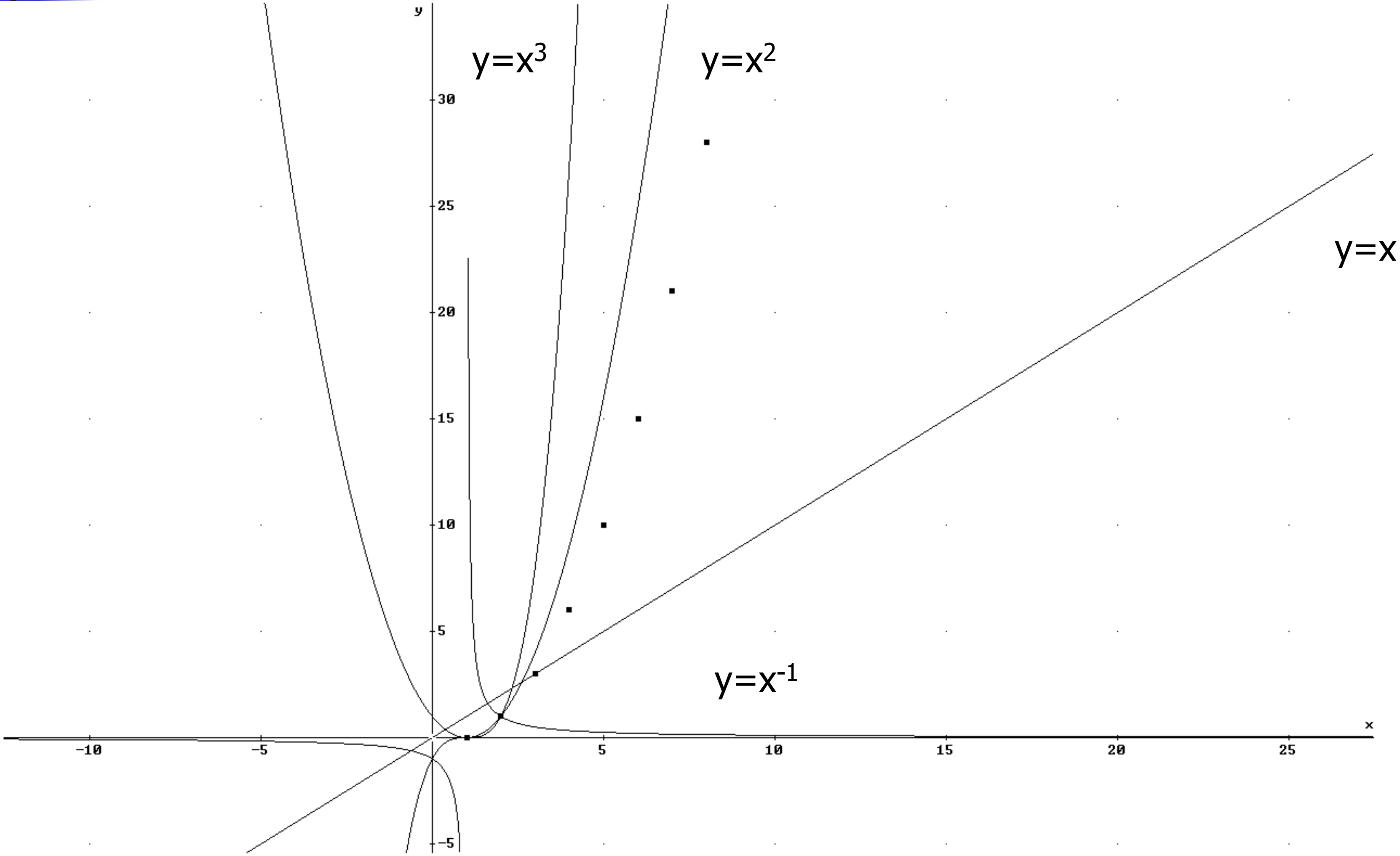
n	e(n)	1st Difference
1	0	
		1 – 0 = 0
2	1	
		3- 1 = 2
3	3	
		6 - 3 = 3
4	6	
		10 – 6 = 4
5	10	
		15 – 10 = 5
6	15	
		21 – 15 = 6
7	21	
		28 – 21 = 7
8	28	
		?
9	?	

Ladder of Powers

- How do we determine if the data is quadratic or cubic?
- Ladder of Powers is list of power functions $p(x) = Ax^n$ where A=1 and n is an integer.
- Plot power functions with data to determine which power function most closely matches the steepness and curvature of the data.

Ladder of Powers
.   .   .
$p(x) = x^5$
$p(x) = x^4$
$p(x) = x^3$
$p(x) = x^2$
$p(x) = x$
$p(x) = x^{-1}$
$p(x) = x^{-2}$
$p(x) = x^{-3}$
$p(x) = x^{-4}$
$p(x) = x^{-5}$
.   .

Ladder of Powers – Which power function best matches the curvature and steepness of the data?



Finite Differences – Quadratic case

- The model for the edges problem appears to be quadratic. How do we determine the model with finite differences?
  - Find the second successive difference – difference of the first difference.
  - If the second difference is constant the data has a quadratic model.

Finite Differences -Quadratic Case

Compare the data differences table to the finite differences table for the general quadratic case (Table 5, Pg 262). What is the quadratic model for the edges problem?

- Solution:
- $e(n) = \frac{1}{2} n^2 - \frac{1}{2} n$

n	e(n)	1st Difference	2nd Difference
1	0		
		1	
2	1		2 – 1 = 1
		2	
3	3		3 – 2 = 1
		3	
4	6		4 – 3 = 1
		4	
5	10		5 – 4 = 1
		5	
6	15		6 – 5 = 1
		6	
7	21		7 – 6 = 1
		7	
8	28		8 – 7 = 1
		8	
9	36		

Finite Differences Summary

- Useful method if the data is theoretic with no error or has little measurement error and variation.
- If there is any variation we have to look for a difference which is approximately constant.
- Try a finite differences problem where the model is a cubic polynomial. Which finite difference column would be constant?