Cycle Spectra of Hamiltonian Graphs

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Fall 2011 Southeastern Section Meeting of the AMS
Wake Forest University
Winston-Salem, NC
25 September 2011
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**Theorem (Bondy (1971))**

If $d(u) + d(v) \geq n$ whenever $u$ and $v$ are non-adjacent, then $G = K_{n/2,n/2}$ or $S(G) = \{3, \ldots, n\}$.
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Theorem (Gould–Haxell–Scott (2002))

\( \forall \varepsilon > 0 \exists c: \text{ if } G \text{ is a graph with } \delta(G) \geq \varepsilon n \text{ and maximum even cycle length } 2\ell, \text{ then } S(G) \text{ contains all even lengths up to } 2\ell - c. \)
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Theorem (Gould–Haxell–Scott (2002))

$\forall \varepsilon > 0 \exists c$: if $G$ is a graph with $\delta(G) \geq \varepsilon n$ and maximum even cycle length $2\ell$, then $S(G)$ contains all even lengths up to $2\ell - c$.

Conjecture

$\exists c$: if $G$ is a Hamiltonian subgraph of $K_{n,n}$ with $\delta(G) \geq c\sqrt{n}$, then $S(G) = \{4, 6, \ldots, 2n\}$. 
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Conjecture (Erdős)
If $G$ has girth $g$ and average degree $k$, then $s(G) \geq \Omega(k^{\lfloor (g-1)/2 \rfloor})$.

(Sudakov–Verstraëte 2008) True for all $g$.

Question (Jacobson–Lehel)
Lower bounds on $s(G)$ when $G$ is Hamiltonian and $k$-regular.
In particular, what about $k = 3$?
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Example (Jacobson–Lehel)

$S(G) = \{4, 6\} \cup \left\{ \frac{2}{3} n, \frac{2}{3} n + 2, \frac{2}{3} n + 4, \ldots, n \right\}$
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Generalizes to provide $k$-regular Hamiltonian graphs with $s(G) = \frac{k-2}{2k} n + k$ when $2k$ divides $n$. 
How small can the cycle spectrum be?

Definition
Let $f_n(m)$ be the minimum size of the cycle spectrum of an $n$-vertex Hamiltonian graph with $m$ edges.
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Theorem (Bondy (1971))

If $G$ is an $n$-vertex Hamiltonian graph with $m$ edges and $m > \frac{n^2}{4}$, then $G$ is \textit{pancyclic} (has cycles of all lengths from 3 to $n$).
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Theorem (Entringer–Schmeichel (1988))
If $G$ is an $n$-vertex bipartite Hamiltonian graph with $m$ edges and $m > n^2/8$, then $G$ is bipancyclic (has cycles of all even lengths from 4 to $n$).
Overlapping chords lemma

Lemma

Let $G$ be a graph with an $x, y$-path $P$ plus $h$ pairwise-overlapping chords of length $\ell$. Then $G$ contains $x, y$-paths of $h - 1$ distinct lengths. Having only $h - 1$ lengths requires that

1. the chords are consecutive along $P$, and
2. $\ell$ is odd and $h \geq \frac{\ell + 3}{2}$. 

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G: an x, y-path P plus h pairwise-overlapping chords of length ℓ. Then G contains x, y-paths of h − 1 distinct lengths. Having only h − 1 lengths requires that

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- Let $n$ be the length of $P$.
- Let $e_1, \ldots, e_h$ be the chords in $G$. 
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- Let \( n \) be the length of \( P \).
- Let \( e_1, \ldots, e_h \) be the chords in \( G \).
- Let \( P_{i,j} \) be the \( x, y \)-path using \( e_i, e_j \), and edges of \( P \).
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- The length of $P_{i,j}$ is $n + 2 - 2d(e_i, e_j)$. 

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**Proof.**

![Diagram showing overlapping chords]

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$\exists d(e_i, e_j)$
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- The length of \( P_{i,j} \) is \( n + 2 - 2d(e_i, e_j) \).
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- So, there is a chord immediately preceding \(e_j\).
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Proof.

- Lengths of paths: $n, n - 2, \ldots, n - 2(h - 2)$.
- Path with a single chord: length $n - (\ell - 1)$.
- So $\ell - 1 \in \{0, 2, \ldots, 2(h - 2)\}$. 
Greedy chord system

- $G$: Hamiltonian cycle $C$ plus $q$ chords of length $\ell$
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- Find many distinct cycle lengths using a greedy chord system.
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- Choose a forward direction along $C$. 

Diagram:

- $e_1$: chord with most overlapping chords going forward.
- $e_2$: first chord not overlapping $e_1$.
- $e_3$: first chord not overlapping $e_2$ or $e_1$.
- $e_4$: first chord not overlapping $e_3$ or $e_1$. 

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- Also: $|F_1| \geq |F^*|$. 

- $\alpha$
Spectrum bands

We find many cycle lengths by dividing the space of possible cycle lengths into bands.
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Let $uv$ be a chord such that $C[u, v]$ has length $\ell$. Replacing $C[u, v]$ with $uv$ reduces the length of a cycle containing $C[u, v]$ by $\ell - 1$. 
Spectrum bands

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- Let $uv$ be a chord such that $C[u, v]$ has length $\ell$. Replacing $C[u, v]$ with $uv$ reduces the length of a cycle containing $C[u, v]$ by $\ell - 1$.
- We have $\alpha$ bands at the top, each of size $\ell - 1$. 
Spectrum bands

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- The $j$th band: from $n - j(\ell - 1) + 1$ to $n - (j - 1)(\ell - 1)$. 
- The short cycles: lengths below the top $\alpha$ bands. 
- The long cycles: lengths in the top 2 bands.
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Lemma

If $\alpha \geq 2$, then $G$ has short cycles of at least $\frac{|F^*| - 1}{2}$ distinct lengths.
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- We may assume $|F^*| \geq 2$. 

Diagram: 
- $v_{\ell+1}$
- $e_1$
- $v_1$
- $e^*$
- $v_j$
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If $\alpha \geq 2$, then $G$ has short cycles of at least $\frac{|F^*|-1}{2}$ distinct lengths.

- We may assume $|F^*| \geq 2$.
- Consider a chord $e \in F^*$ with $e \neq e^*$.
- Cycle using $e^*$ and $e$ has length $2(k - j + 1)$. 

$e^*$  
$v_k$  
$v_1$  
vj  
v_{\ell+1}$  
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Note: $j \geq 1 + \alpha \ell$. 

This cycle has length at most $n - \alpha \ell + 2$. 

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Diagram: A graph with vertices $v_1, v_2, \ldots, v_{\ell+1}$ and edges $e_1, e_2, e_3, e_4, e_\alpha, e^*$ connecting them.
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We obtain $|F^* - 1|$ short cycles.
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If $\alpha \geq 2$, then $G$ has short cycles of at least $\frac{|F^*| - 1}{2}$ distinct lengths.

- We obtain $|F^* - 1|$ short cycles.
- Each length occurs at most twice.
Longer cycles
Longer cycles

<table>
<thead>
<tr>
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<td>( \alpha )</td>
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Longer cycles

\[ \alpha \]

Short Cycles

Long Cycles

\[ \ell > 3 \]

\[ e_1, e_2, e_3, e_4, e_\alpha, e^* \]
A long cycle is good if it contains $C[u, v]$.
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- Let $\rho$ be the number of lengths of good cycles.
- Overlapping chords lemma: $\rho \geq |F_1| - 1$.

First, suppose $\rho \geq |F_1|$. 

Longer cycles

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Short Cycles

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- $u$ and $v$ are vertices.
- $e_1$, $e_2$, $e_3$, $e_4$, and $e_\alpha$ are edges.

Each length occurs at most twice. Add one more set. Now: $\alpha$ sets of $\rho$ lengths; each length appears at most once.
Using a chord shifts these lengths down by \( \ell - 1 \).

This yields \( \alpha - 1 \) sets of \( \rho \) lengths. Each length occurs at most twice.
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Longer cycles

Short Cycles

Long Cycles

\[ \frac{\alpha \rho}{2} \]

So we have \( \frac{\alpha \rho}{2} \) longer cycle lengths, plus \( \frac{|F^*| - 1}{2} \) short cycle lengths.
Longer cycles

\[ \alpha \]

Short Cycles

Long Cycles

\[ \ell - 1 \]

\[ u \]

\[ v \]

\[ e_1 \]

\[ e_2 \]

\[ e_3 \]

\[ e_4 \]

\[ e_\alpha \]

\[ \alpha \rho \]

\[ |F^*| - 1 \]

\[ s(G) \geq \frac{\alpha}{2} |F_1| + \frac{|F^*| - 1}{2} \]

\[ \geq \frac{\alpha}{2} q - \frac{|F^*|}{\alpha} + \frac{|F^*| - 1}{2} \]

\[ \geq \frac{q - 1}{2} \]

So we have \( \frac{\alpha \rho}{2} \) longer cycle lengths, plus \( \frac{|F^*| - 1}{2} \) short cycle lengths.

Since \( \rho \geq |F_1| \),

\[ s(G) \geq \frac{\alpha |F_1|}{2} + \frac{|F^*| - 1}{2} \]

\[ \geq \frac{\alpha q - |F^*|}{2} + \frac{|F^*| - 1}{2} \]

\[ \geq \frac{q - 1}{2} \]
Longer cycles

\[ u \quad \text{Short Cycles} \quad \alpha \quad \text{Long Cycles} \]

\[
\begin{array}{c|c|c|c}
\alpha & \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot \\
3 & 2 & 1 \\
\end{array}
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\[ \ell - 1 \]

\[ \text{Otherwise } \rho = |F_1| - 1. \]
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$\ell - 1$

$\Rightarrow$ Otherwise $\rho = |F_1| - 1$.

$\Rightarrow$ The Overlapping cycles lemma implies:
### Longer cycles

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Ηα = \|F_1\| - 1.

- Otherwise ρ = |F_1| - 1.

- The Overlapping cycles lemma implies:
  1. \( ι \) is odd
  2. Chords in \( F_1 \) are consecutive
  3. \(|F_1| \geq (ι + 3)/2\)
Longer cycles

\[ \alpha \]

Short Cycles

Long Cycles

\[ \ell - 1 \]

\( u \) \( v \)

- Otherwise \( \rho = |F_1| - 1. \)
- The Overlapping cycles lemma implies:
  1. \( \ell \) is odd
  2. Chords in \( F_1 \) are consecutive
  3. \( |F_1| \geq (\ell + 3)/2 \)
- We exploit the structure in two cases to show

\[ s(G) \geq \left( q - 1 - \frac{q}{\ell} \right)/2. \]
Summary and Open Problems

Theorem

If $G$ is an $n$-vertex Hamiltonian graph with $p$ chords, then

$$s(G) \geq \sqrt{p} - \frac{1}{2} \ln p - 1.$$

Open Problems

▶ What is the maximum number of edges in an $n$-vertex bipartite Hamiltonian graph that is not bipancyclic? The answer lies between $(1 + o(1))n^2/16$ and $(1 + o(1))n^2/8$.

▶ What is the maximum number of edges in an $n$-vertex Hamiltonian graph with $s(G) < n/2 - 1$? The answer lies between $(1 + o(1))n^2/16$ and $(1 + o(1))n^2/4$.

▶ Obtain better bounds on $f_n(m)$.

▶ Is a constant $c$ such that $s(G) \geq cn$ for every Hamiltonian graph $G$ with $\delta(G) \geq 3$?
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Thank You