Abstract

The altitude of a graph $G$, denoted $f(G)$, is the largest integer $k$ such that under each ordering of $E(G)$, there exists a path of length $k$ which traverses edges in increasing order. In 1971, Chvátal and Komlós asked for $f(K_n)$, where $K_n$ is the complete graph on $n$ vertices. In 1973, Graham and Kleitman proved that $f(K_n) \geq \sqrt{n - 3/4} - 1/2$ and in 1984, Calderbank, Chung, and Sturtevant proved that $f(K_n) \leq (\frac{1}{2} + o(1)) n$. We show that $f(K_n) \geq (\frac{1}{20} - o(1))(n/\lg n)^{2/3}$.

1 Introduction

A totally ordered graph is a graph $G$ that is associated with a total ordering of its vertex set $V(G)$ and a total ordering of its edge set $E(G)$. We use $T(G)$ and $T'(G)$ to denote the total orderings of $V(G)$ and $E(G)$ respectively. When only the vertices or only the edges of $G$ are totally ordered, we call $G$ an ordered graph or an edge-ordered graph, respectively. An ordering, edge-ordering, or total ordering of a graph $G$ is an ordered, edge-ordered, or totally ordered graph whose underlying graph is $G$.

In an edge-ordered graph $G$, a monotone path is a path which traverses edges in increasing order with respect to $T'(G)$. A monotone trail is similar, except that a trail is allowed to revisit vertices. The altitude of a graph $G$, denoted $f(G)$, is the maximum integer $k$ such that every edge-ordering of $G$ contains a monotone path of length $k$. Also, let $f^*(G)$ be the maximum integer $k$ such that every edge-ordering of $G$ contains a monotone trail of length $k$.

In 1971, Chvátal and Komlós [4] asked for $f(K_n)$ and $f^*(K_n)$, where $K_n$ denotes the complete graph on $n$ vertices. Citing private communication, Chvátal and Komlós noted in their 1971 paper that Graham and Kleitman had already proved $\Omega(n^{1/2}) \leq f(K_n) < (\frac{3}{4} + \varepsilon)n$ and established $f^*(K_n)$ exactly: $f^*(K_n) = n - 1$ unless $n \in \{3, 5\}$, in which case $f^*(K_n) = n$.

To show $f^*(K_n) \geq n - 1$, Graham and Kleitman [6] proved that if $G$ has average degree $d$, then $f^*(G) \geq d$. Friedgut communicated to Winkler [13] an elegant formulation of their proof, known as the pedestrian argument. For an $n$-vertex edge-ordered graph $G$, the pedestrian argument involves $n$ pedestrians, with one starting at each vertex in $G$. An announcer calls out the names of the edges in order according to $T'(G)$. When $e$ is called, both pedestrians at the endpoints of $e$ traverse $e$, trading places. Since each pedestrian travels along a monotone trail and each edge is traversed by two pedestrians, the average length of a pedestrian’s monotone trail is $2|E(G)|/n$, which equals $d$. The pedestrian argument has recently been modified to produce monotone paths (see [8] and [5]).
Determining the altitude of a graph appears to be difficult in general. In 1973, Graham and Kleitman [3] published their results on \( f(K_n) \) and \( f^*(G) \). In particular, they proved that 
\[
\sqrt{n - 3/4} - 1/2 \leq f(K_n) < 3n/4,
\]
and they conjectured that \( f(K_n) \) is closer to their upper bound than their lower bound. They also commented that, with additional effort, their lower bound could be improved to 
\[
f(K_n) \geq (c - o(1)) \sqrt{n}
\]
for some \( c > 1 \). In his Master’s thesis from the same year, Rödl [11] proved that if \( G \) has average degree \( d \), then 
\[
f(G) \geq (1 - o(1)) \sqrt{d};
\]
for \( G = K_n \), Rödl’s result matches the Graham–Kleitman lower bound asymptotically. Rödl also noticed that the ideas in the Graham–Kleitman upper bound can be combined with results in design theory to prove 
\[
f(K_n) \leq \left( \frac{7}{12} + o(1) \right) n.
\]
In 1984, Calderbank, Chung, and Sturtevant [3] obtained the best known upper bound: 
\[
f(K_n) \leq \left( \frac{1}{2} + o(1) \right) n.
\]
After 1984, explicit progress on determining \( f(K_n) \) slowed (but see [2] for exact values for \( n \leq 8 \)). In the meantime, other interesting results on the altitude of graphs have appeared.

In 2001, Roditty, Shoham, and Yuster [10] proved that \( f(G) \leq 9 \) if \( G \) is planar and showed that 
\[
f(C_n \vee K_2) \geq 5 \text{ for } n \geq 99,
\]
where \( C_n \vee K_2 \) is the planar graph obtained by joining the \( n \)-vertex cycle \( C_n \) and a pair of non-adjacent vertices. Consequently, the maximum of altitude of a planar graph is between 5 and 9.

Clearly, \( f(G) \leq f^*(G) \). The \textit{edge-chromatic number} of \( G \), denoted \( \chi'(G) \), is the minimum \( k \) such that \( E(G) \) is the union of \( k \) matchings. Ordering \( E(G) \) so that each matching is an interval shows that 
\[
f^*(G) \leq \chi'(G).
\]
Vizing’s theorem [12] states that \( \chi'(G) \leq \Delta(G) + 1 \), where \( \Delta(G) \) is the maximum degree of \( G \). It follows that 
\[
f(G) \leq \Delta(G) + 1.
\]

Improving a result of Yuster [14], Alon [11] gave a short proof that there exist \( k \)-regular graphs \( G \), with \( f(G) \geq k \), as follows. The \textit{girth} of \( G \) is the length of a shortest cycle in \( G \). If \( G \) has girth \( g \), then every trail of length less than \( g \) is a path. Therefore 
\[
f(G) \geq \min\{ g - 1, f^*(G) \} \geq \min\{ g - 1, d \},
\]
where \( d \) is the average degree of \( G \). In particular, if \( G \) is \( k \)-regular and has girth larger than \( k \), then 
\[
f(G) \geq k.
\]
For \( k = 3 \), better constructions are known. Mynhardt, Burger, Clark, Falvai, and Henderson [9] characterized the 3-regular graphs with girth at least 5 and altitude 3, and then used the characterization to show that the flower snarks are examples of 3-regular graphs with altitude 4. For \( k \geq 4 \), it remains open to decide whether there are graphs \( G \) with \( \Delta(G) = k \) and \( f(G) = k + 1 \).

A \textit{Hamiltonian path} in a graph is a path containing all of its vertices. Katrenič and Semanišin [7] proved that deciding whether a given edge-ordered graph contains a Hamiltonian monotone path is NP-complete. Although it seems likely that computing the altitude of a given graph is NP-hard or worse, we note that the result of Katrenič and Semanišin does not directly imply this.

Lavrov and Loh [8] investigated the maximum length of a monotone path in a random edge-ordering of \( K_n \). They showed that with probability tending to 1, a random edge-ordering of \( K_n \) contains a monotone path of length at least \( 0.85n \). Consequently, edge-orderings of \( K_n \) that give sublinear upper bounds on \( f(K_n) \), if they exist, are rare. They also proved that with probability at least \( 1/e - o(1) \), a random edge-ordering of \( K_n \) contains a Hamiltonian monotone path. The common strengthening of these results leads to a natural and beautiful conjecture.

**Conjecture 1.1** (Lavrov–Loh [8]). With probability tending to 1, a random edge-ordering of \( K_n \) contains a Hamiltonian monotone path.

Recently, De Silva, Molla, Pfender, Retter, and Tait [5] proved that 
\[
f(Q_n) \geq n/\lg n
\]
where \( Q_n \) is the \( n \)-dimensional hypercube and \( \lg \) denotes the base-2 logarithm. They also showed that if

\[
2 \text{ Monotone Path Algorithm}
\]

simple algorithm to extend monotone paths.

Let \( \sqrt{n} \) be a positive integer and \( K \) be a totally ordered graph.

Proof. Let \( A \) be the index of the row in \( A \) containing the vertex outside \( x_{k-1} \).

Extending a given monotone path is a key step in our algorithm. The height of a nontrivial monotone path \( x_0 \ldots x_k \) is the height of its last edge \( x_{k-1}x_k \).

Lemma 2.1. Let \( G \) be a totally ordered graph. For \( 1 \leq k < r \), each monotone path of length \( k \) and height \( r \) extends to a monotone path of length \( k+1 \) and height at least \( r-k \).

Proof. Let \( A \) be the height table of \( G \), and let \( x_0 \ldots x_k \) be a monotone path of length \( k \) and height \( r \). Since \( x_{k-1}x_k \) appears in row \( r \) in \( A \), this edge did not already appear when \( A(i,x_k) \) is defined for \( i < r \). It follows that for \( i < r \), the cell \( A(i,x_k) \) contains an edge incident to \( x_k \) which is larger than \( x_{k-1}x_k \) in \( T'(G) \). Let \( A(i,x_k) : r-k \leq i \leq r-1 \). Since \( |S| = k \) and \( x_{k-1}x_k \notin S \), some edge in \( S \) joins \( x_k \) with a vertex outside \( \{x_0, \ldots, x_{k-1}\} \) and extends the path as claimed. \( \square \)

Starting with a single edge and iterating Lemma 2.1, we obtain the following.

Lemma 2.2. Let \( G \) be a totally ordered graph and let \( x_0x_1 \) be an edge in \( G \) of height \( r \). If \( t \) is a positive integer and \( \left( \frac{3}{2} \right) < r \), then \( G \) contains monotone path \( x_0x_1 \ldots x_t \) of height at least \( r - \left( \frac{3}{2} \right) \).
Proof. By induction on $t$. The lemma is clear when $t = 1$. For $t > 1$, the inductive hypothesis implies that $G$ contains a monotone path $x_0x_1 \ldots x_{t-1}$ of height at least $r - (\binom{t-1}{2})$. With $k = t - 1$, we apply Lemma 2.1 to obtain a monotone path $x_0 \ldots x_t$ with height at least $(r - (\binom{t-1}{2})) - (t - 1)$ which equals $r - \binom{t}{2}$.

Using Lemma 2.2, we match Rödl’s bound $f(G) \geq (1 - o(1))\sqrt{d}$ asymptotically. We include the short proof for completeness.

**Theorem 2.3.** If $G$ has average degree $d$, then $f(G) \geq \lfloor 1/2 + \sqrt{d} \rfloor$.

**Proof.** Let $H$ be a total ordering of $G$, and let $x_0x_1$ be an edge of maximum height $r$. Since each row of the height table contains $n$ cells, it follows that $r \geq |E(G)|/n = d/2$. If $t$ is a positive integer and $\binom{t}{2} < d/2$, then we apply Lemma 2.1 to obtain a monotone path of length $t$ in $H$. Hence, $\binom{t}{2} < d/2$ implies that $f(G) \geq t$. With $t = \lfloor 1/2 + \sqrt{d} \rfloor$, we have that $\binom{t}{2} < d/2$ and therefore $f(G) \geq \lfloor 1/2 + \sqrt{d} \rfloor$.

Let $G$ be a totally ordered graph and let $x_0 \ldots x_k$ be a monotone path in $G$. Viewing height as a resource, extending $x_0 \ldots x_k$ becomes more expensive as $k$ grows. When extending becomes too expensive, we delete $\{x_0, \ldots, x_{k-2}\}$ from $G$ to form a new totally ordered graph $G'$ (which inherits the orderings of $V(G)$ and $E(G)$), and we extend $x_{k-1}x_k$ to a monotone path in $G'$. For this to work, we must show that the height of $x_{k-1}x_k$ does not decrease too much when we delete $\{x_0, \ldots, x_{k-2}\}$ from $G$.

**Definition 2.4.** Let $G$ be a totally ordered graph. For $S \subseteq V(G)$ and an edge $e$ in $G - S$, we define $\text{drop}(G, S, e)$ to be $h_G(e) - h_{G-S}(e)$. For $s \leq n - 2$, let $g(n, s)$ be the maximum of $\text{drop}(G, S, e)$ over all $n$-vertex totally ordered graphs $G$, all sets $S$ of $s$ vertices in $G$, and all edges $e \in E(G - S)$.

Note that $g(n, s)$ is monotonic in $n$, since adding isolated vertices to a totally ordered graph $G$ and inserting them arbitrarily into the vertex ordering gives a larger totally ordered graph $G'$ such that $\text{drop}(G, S, e) = \text{drop}(G', S, e)$ for all $S \subseteq V(G)$ and $e \in E(G - S)$.

**Lemma 2.5.** Let $G$ be an $n$-vertex totally ordered graph and let $x_0x_1$ be an edge of height $r$. If $s$ is a positive integer and $s \leq n - 2$, then $G$ contains a monotone path extending $x_0x_1$ of length at least $sk + 1$, where $k = \lfloor (r - 1)/(\binom{s+1}{2} + g(n, s)) \rfloor$.

**Proof.** By induction on $n$. If $k = 0$, then the lemma is clear. Otherwise, $r - 1 \geq \binom{s+1}{2} + g(n, s)$ and we may apply Lemma 2.2 to obtain a monotone path $x_0 \ldots x_{s+1}$ of height at least $r - \binom{s+1}{2}$. Let $S = \{x_0, \ldots, x_{s-1}\}$ and let $G' = G - S$. We have that $h_{G'}(x_{s}s_{s+1}) = h_G(x_{s}s_{s+1}) - \text{drop}(G, S, x_{s}s_{s+1}) \geq r - (\binom{s+1}{2}) - g(n, s)$.

Applying the inductive hypothesis to $G'$ and $x_{s}s_{s+1}$, we obtain a monotone path $P'$ in $G'$ extending $x_{s}s_{s+1}$ of length at least $sk' + 1$, where

$$k' = \left\lfloor \frac{r - (\binom{s+1}{2}) - g(n, s) - 1}{(\binom{s+1}{2}) + g(n, s)} \right\rfloor \geq \left\lfloor \frac{r - (\binom{s+1}{2}) - g(n, s) - 1}{(\binom{s+1}{2}) + g(n, s)} \right\rfloor = k - 1.$$

Prepending $x_0 \ldots x_s$ to $P'$ produces a monotone path in $G$ of length at least $s + sk' + 1$, and $s + sk' + 1 \geq sk + 1$.
3 The Token Game

Our goal is to prove an upper bound on \( g(n,s) \). Let \( G \) be an \( n \)-vertex totally ordered graph, and let \( S \) be a set of \( s \) vertices of \( G \). We analyze an iterative process which obtains the height table of \( G - S \) from the height table of \( G \). Let \( G' = G - S \), let \( A \) be the array obtained from the height table of \( G \) by deleting columns indexed by vertices in \( S \), and let \( A' \) be the height table of \( G' \). Note that the cells of both \( A \) and \( A' \) are indexed by \( Z \), where \( Z = \{1,2,3,\ldots\} \times V(G') \). We order \( Z \) in the same order as the corresponding cells in \( A' \) are defined; that is, \((i,u) \leq (i',v)\) if and only if \( i < i' \) or \( i = i' \) and \( u \leq v \) in \( T(G') \). For \( \beta \in Z \), the open down-set of \( \beta \), denoted \( D(\beta) \), is \( \{\gamma \in Z : \gamma < \beta\} \) and the closed up-set of \( \beta \), denoted \( U[\beta] \), is \( \{\gamma \in Z : \gamma \geq \beta\} \). Similarly, the interval \([\beta, \gamma]\) is \( \{\delta \in Z : \beta \leq \delta \leq \gamma\} \).

We produce a sequence of arrays \( \{A_\beta : \beta \in Z\} \) which initially resemble \( A \) and later resemble \( A' \). For \( \beta \in Z \), the cells of \( A_\beta \) are indexed by \( Z \) and are partitioned into a lower part indexed by \( D(\beta) \) and an upper part indexed by \( U[\beta] \).

For \( \beta \in Z \), each cell in \( A_\beta \) is either empty, contains an edge in \( G' \), or contains an object called a hole. Moreover, each edge in \( G' \) appears in one cell in \( A_\beta \). Each \( A_\beta \) also has a critical interval \([i,u),(j,u)\] where \( \beta = (i,u) \) and \( j \) is the least integer such that \( j \geq i \) and \( A_\beta(j,u) \) does not contain a hole.

**Lemma 3.1.** There is a sequence of arrays \( \{A_\beta: \beta \in Z\} \) such that each column in the initial array has at most \( s \) holes, and for each \( \beta \in Z \) the following hold.

1. If \( \delta < \beta \), then \( A_\beta(\delta) = A'(\delta) \).
2. If \( \delta \geq \beta \) and \( A_\beta(\delta) \) does not contain a hole, then \( A_\beta(\delta) = A(\delta) \).
3. If \( \gamma \) is the successor of \( \beta \) in \( Z \), then \( A_\gamma \) is obtained from \( A_\beta \) by swapping \( A_\beta(\beta) \) and \( A_\beta(\delta) \), where \( \delta \) is in the critical interval of \( A_\beta \). Moreover, if \( \beta \) and \( \delta \) index cells in distinct columns \( u \) and \( v \), then \( A_\beta(\delta) = uv \).

**Proof.** Recall that \( A \) is obtained from the height table of \( G \) by deleting columns indexed by vertices in \( S \). Note that \( A \) omits every edge with both endpoints in \( S \) and contains every edge in \( G' \). An edge \( uv \in [S,S] \) with \( u \not\in S \) and \( v \in S \) appears in \( A \) if and only if \( uv \) is in column \( u \) in the height table of \( G \). Let \( \alpha \) be the minimum element in \( Z \), and let \( A_\alpha \) be the array obtained from \( A \) by replacing edges in \( [S,S] \) with holes. If \( u \) indexes a column in \( A_\alpha \), then each hole in column \( u \) replaces an edge \( uv \) in \( G \) with \( v \in S \), and therefore each column in \( A_\alpha \) contains at most \( s \) holes. Clearly, every edge in \( G' \) appears once in \( A_\alpha \), and \( A_\alpha \) satisfies properties (1) and (2).

We obtain other arrays iteratively. Let \( \beta = (i,u) \), let \( \gamma \) be the successor of \( \beta \), and suppose that \( A_\beta \) has been previously defined but \( A_\gamma \) is not yet defined. Analogously to \( A_\beta \), we partition the cells of \( A' \) into a lower part indexed by \( D(\beta) \) and an upper part indexed by \( U[\beta] \). Since \( A_\beta \) and \( A' \) contain the same set of edges and agree on their lower parts, it follows that the upper parts of \( A_\beta \) and \( A' \) contain the same edges (possibly in a different order). We consider two cases, depending on whether \( A'(\beta) \) is empty or contains an edge in \( G' \).

**Case 1:** \( A'(\beta) \) is not empty. Let \( e = A'(\beta) \), and let \( \delta \) be the index of the cell in \( A_\beta \) containing \( e \). We claim that \( \delta \) is in the critical interval \([i,u),(j,u)\] of \( A_\beta \). Since \( e \) is in the upper part of \( A' \), it follows that \( e \) is in the upper part of \( A_\beta \) and so \( \delta \geq \beta = (i,u) \). Since \( \delta, (j,u) \in U[\beta] \) and neither \( A_\beta(\delta) \) nor \( A_\beta(j,u) \) contains a hole, it follows from (2) that \( A(\delta) = A_\beta(\delta) = e \) and \( A(j,u) = A_\beta(j,u) \). Suppose for a contradiction that \( \delta > (j,u) \). Note that \( e \) is available for \( A(j,u) \)
when building the height table of $G$, and so $A(j, u) = e'$ for some edge $e'$ incident to $u$ such that $e' > e$ in $T'(G)$. Since $A_\beta(j, u) = A(j, u) = e'$, it follows that both $e$ and $e'$ appear in the upper part of $A_\beta$ and hence in the upper part of $A'$ also. Therefore both $e$ and $e'$ are available for $A'(\beta)$ when building the height table of $G'$. The selection of $e$ over $e'$ for $A'(\beta)$ implies that $e > e'$ in $T'(G')$, contradicting that $e' > e$ in $T'(G)$. Therefore $\delta \leq (j, u)$ and $\delta$ is in the critical interval of $A_\beta$ as claimed. Obtain $A_\gamma$ from $A_\beta$ by swapping the contents of cells $A_\beta(\beta)$ and $A_\beta(\delta)$ (if $\beta = \delta$, then $A_\gamma = A_\beta$). Note that if $\delta$ indexes a cell in column $v$ and $v \neq u$, then $A'(\beta) = e$ and $A(\delta) = e$ imply that $e$ is incident to both $u$ and $v$, so that $A_\beta(\delta) = e = uv$, satisfying (3).

We check that $A_\gamma$ satisfies (1) and (2). Since $\gamma$ is the successor of $\beta$ and $A_\gamma(\beta) = A_\beta(\delta) = e = A'(\beta)$, it follows that $A_\gamma$ satisfies (1). If the critical interval $[(i, u), (j, u)]$ of $A_\beta$ has size 1, then $\beta = (i, u) = \delta = (j, u)$ and $A_\gamma = A_\beta$, implying that $A_\gamma$ satisfies (2). Otherwise $j > i$ and $A_\beta(\beta)$ contains a hole. Relative to $A_\beta$, the only change in the upper part of $A_\gamma$ is that $A_\gamma(\delta)$ becomes a hole after swapping $A_\beta(\beta)$ and $A_\beta(\delta)$, and so $A_\gamma$ satisfies (2).

Case 2: $A'(\beta)$ is empty. This implies that the upper part of $A'$ contains no edge incident to $u$, and so the upper part of $A_\beta$ also contains no edge incident to $u$. In particular, $A_\beta(j, u)$ is empty, where $[(i, u), (j, u)]$ is the critical interval of $A_\beta$. We obtain $A_\gamma$ from $A_\beta$ by swapping the contents of cells $A_\beta(i, u)$ and $A_\beta(j, u)$, satisfying (3). Since $A_\gamma(\beta)$ and $A'(\beta)$ are both empty, $A_\gamma$ satisfies (1). Relative to $A_\beta$, the upper part of $A_\gamma$ is either unchanged or contains a new hole at $A_\gamma(j, u)$. It follows that $A_\gamma$ also satisfies (2).

Given the sequence of arrays $\{A_\beta: \beta \in Z\}$ from Lemma 3.1, we obtain a useful upper bound on $\text{drop}(G, S, e)$.

**Lemma 3.2.** Let $e$ be an edge in $G'$ and choose $\beta \in Z$ so that $A'(\beta) = e$. If $[(i, u), (j, u)]$ is the critical interval of $A_\beta$, then $\text{drop}(G, S, e) \leq j - i$.

**Proof.** Since $\beta = (i, u)$ and $e$ appears in row $i$ of the height table of $G'$, it follows that $h_{G'}(e) = i$. Let $\delta$ index the cell in $A_\beta$ containing $e$. Since the successor $A_\gamma$ of $A_\beta$ satisfies $A_\gamma(\beta) = A'(\beta) = e$, it follows that $A_\gamma$ is obtained from $A_\beta$ by swapping $A_\beta(\beta)$ with $A_\beta(\delta)$. By (3), we have that $\delta$ is in the critical interval $[(i, u), (j, u)]$ of $A_\gamma$, and so $\delta = (\ell, v)$ where $i \leq \ell \leq j$. Since $\delta \geq \beta$ and $A_\beta(\delta)$ is not a hole, by (2) we have that $e = A_\beta(\delta) = A(\delta)$. Therefore $e$ appears in row $\ell$ of the height table of $G$ and so $h_G(e) = \ell$. We conclude $\text{drop}(G, S, e) = \ell - i \leq j - i$. 

We define the **height** of a critical interval $[(i, u), (j, u)]$ to be $j - i$. Note that the height of the critical interval of $A_\beta$ is at most the number of holes in column $u$ of $A_\beta$. Also, by property (1) of Lemma 3.1, all holes of $A_\beta$ are contained in the upper part of $A_\beta$. Analyzing the movement of the holes as $\beta$ increases in $Z$ naturally leads to a single player game.

A **token game** is a game played on an array $B$ with rows indexed by the positive integers and columns indexed by a finite list. Let $B(i, u)$ denote the cell in row $i$ and column $u$. Each cell in $B$ is empty or contains a **token**. A token in cell $B(i, u)$ is **grounded** if all cells in column $u$ below $B(i, u)$ contain tokens; a token which is not grounded is **ungrounded**. One of the columns is distinguished as the **active column**.

A step in a token game modifies $B$ to produce a new array $B'$, subject to certain rules. Let $u$ be the active column. If column $u$ contains grounded tokens, then the player may optionally move the highest grounded token in column $u$ from its cell $B(i, u)$ to an empty cell $B(i', v)$, provided that $i' \leq i$ and no prior step in the game moved a token between columns $u$ and $v$. Next, all ungrounded tokens in column $u$ shift down by one cell, and the active column advances cyclically. A step in
which a token moves between columns is a transfer step. The list of arrays produced in a token game is its transcript.

An \((n, s)\)-token game is a token game with \(n\) columns, each of which initially contains at most \(s\) tokens. Let \(\hat{g}(n, s)\) be the maximum number of tokens that can be placed in a single column in an \((n, s)\)-token game. The following gives the connection between \(g(n, s)\) and \(\hat{g}(n, s)\).

**Lemma 3.3.** \(g(n, s) \leq \hat{g}(n - s, s)\)

**Proof.** Let \(G\) be an \(n\)-vertex totally ordered graph and let \(S\) be a set of \(s\) vertices in \(G\) such that \(\text{drop}(G, S, e) = g(n, s)\) for some edge \(e\) in \(G - S\). Let \(G' = G - S\), let \(A'\) be the height table of \(G'\), obtain \(A\) from the height table of \(G\) by deleting columns indexed by \(S\), and apply Lemma 3.1 to obtain the sequence of arrays \(\{A_\beta : \beta \in Z\}\). We use this sequence to play the \((n - s, s)\)-token game so that at least \(g(n, s)\) tokens are placed in some column.

Construct a sequence \(\{B_\beta : \beta \in Z\}\) of token arrays as follows. Let \(\beta = (i, u)\). We put a token in \(B_\beta(j, v)\) if and only if \(A_\beta(k, v)\) contains a hole, where \(k = j + i\) if \(v < u\) in \(T(G')\) and \(k = j + i - 1\) otherwise. Equivalently, we obtain \(B_\beta\) from \(A_\beta\) by removing all edges so that only holes and empty cells remain, shifting cells down to discard the lower part of \(A_\beta\), and replacing holes with tokens.

We claim that the sequence \(\{B_\beta : \beta \in Z\}\) is the transcript of an \((n - s, s)\)-token game in which the active column of \(B_\beta\) is the second coordinate in \(\beta\). Let \(\alpha\) be the minimum element in \(Z\), and note that each column in \(A_\alpha\) contains at most \(s\) holes by Lemma 3.1. It follows that each column in \(B_\alpha\) contains at most \(s\) tokens, satisfying the initial condition of an \((n - s, s)\)-token game.

Let \(\beta = (i, u)\) and let \(\gamma\) be the successor of \(\beta\). From property (3) of Lemma 3.1 we have that \(A_\gamma\) is obtained from \(A_\beta\) by swapping \(A_\beta(\beta)\) and \(A_\beta(\delta)\) for some \(\delta\) in the critical interval \([ (i, u), (j, u) ]\) of \(A_\beta\). If the critical interval has size 1, then \(A_\gamma = A_\beta\) and column \(u\) of \(B_\beta\) contains no grounded tokens. We obtain \(B_\gamma\) from \(B_\beta\) by allowing the tokens in column \(u\) to shift down by 1 cell. The active column advances, completing a legal move in the token game.

Otherwise \(j > i\). Recall that the cells of \(B_\beta\) correspond to the upper part of \(A_\beta\). The cells indexed by the critical interval \([ (i, u), (j, u) ]\) of \(A_\beta\) correspond to the cells in \(B_\beta\) of height at most \(j - i\), except that the last cell \(A_\beta(i, u)(j, u)\) corresponds to \(B_\beta(j - i + 1, u)\) which has height \(j - i + 1\).

Since \(A_\beta(\ell, u)\) contains a hole for \(i \leq \ell < j\), it follows that \(B_\beta(\ell, u)\) contains a grounded token for \(1 \leq \ell \leq j - i\). Since \(A_\beta(\beta)\) contains a hole and \(A_\beta(\delta)\) does not, it follows that \(\delta > \beta\) and we obtain \(B_\gamma\) from \(A_\beta\) by swapping the contents of distinct cells \(A_\beta(\beta)\) and \(A_\beta(\delta)\). Therefore we obtain \(B_\gamma\) from \(B_\beta\) by firstly moving the grounded token in \(B_\beta(1, u)\) to an empty cell of height at most \(j - i\) or to \(B_\beta(j - i + 1, u)\) and secondly shifting the contents of all cells in column \(u\) down by 1 cell. Equivalently, we obtain \(B_\gamma\) from \(B_\beta\) by optionally moving the highest grounded token from \(B_\beta(j - i, u)\) to an empty cell of height at most \(j - i\) and shifting the ungrounded tokens in column \(u\) down by 1 cell. This is allowed in a token game provided that we have not executed a transfer step between a pair of columns more than once.

Suppose that the transition from \(B_\beta\) to \(B_\gamma\) represents the first transfer step between distinct columns \(u\) and \(v\); we may assume without loss of generality that a token is moved from column \(u\) in \(B_\beta\) to column \(v\) in \(B_\gamma\). It follows that a hole in \(A_\beta(\beta)\) is swapped with the contents of \(A_\beta(\delta)\) to form \(A_\gamma\), where \(\beta\) and \(\delta\) index cells in columns \(u\) and \(v\) respectively. By property (3) of Lemma 3.1 we have that \(A_\beta(\delta) = uv\). Since \(\delta > \beta\), the edge \(uv\) is in the upper part of \(A_\beta\). On the other hand, we have \(\beta < \gamma\) and \(A_\gamma(\beta) = A_\beta(\delta) = uv\), and so \(uv\) is in the lower part of \(A_\gamma\). In fact, \(A_{\gamma'}(\beta) = A_\gamma(\beta) = uv\) for \(\gamma' \geq \gamma\), and so \(uv\) is in the lower part of \(A_{\gamma'}\) for all \(\gamma' \geq \gamma\). It follows that there are no subsequent transfer steps between columns \(u\) and \(v\).
Therefore \( \{ B_\gamma : \gamma \in \mathbb{Z} \} \) is the sequence of arrays in an \((n - s, s)\)-token game. Let \( e \) be an edge in \( G' \) with \( \text{drop}(G, S, e) = g(n, s) \), and let \( \beta \) be the index of the cell in \( A' \) containing \( e \). By Lemma 3.2, we have that \( g(n, s) = \text{drop}(G, S, e) \leq j - i \), where \( [(i, u), (j, u)] \) is the critical interval of \( A_\beta \). Since \( A_\beta(\ell, u) \) contains a hole for \( i \leq \ell < j \), it follows that \( B_\beta(\ell, u) \) contains a grounded token for \( 1 \leq \ell \leq j - i \). Hence, it is possible to place at least \( j - i \) tokens in some column in an \((n - s, s)\)-token game and so \( \hat{g}(n - s, s) \geq j - i \).

\[
\text{Proof.} \]

It remains to analyze the \((n, s)\)-token game. Our main tool is to show that in an \((n, s)\)-token game in which the number of tokens in a particular column grows substantially, it is possible to find a subgame with half the number of transfer steps in which a column gains a substantial number of tokens. Eventually, we obtain a contradiction since the number of tokens in a column cannot grow by more than the total number of transfer steps.

**Lemma 3.4.** Let \( B_0, \ldots, B_k \) be the transcript of a token game with a total of \( m \) tokens and at most \( 2^\ell \) transfer steps. Suppose that some column initially contains a tokens in \( B_0 \) and ends with \( b \) tokens in \( B_k \). If \( a' \) and \( r \) are integers such that \( m < (a' + 1)r/2 \), then some subinterval of \( B_0, \ldots, B_k \) is the transcript of a token game with \( m \) tokens and at most \( 2^{\ell - 1} \) transfer steps, in which some column initially has at most \( a' \) tokens and ends with at least \( b' \) tokens, where \( b' = b - a - r + 1 \).

**Proof.** Choose \( j \) so that both \( B_0, \ldots, B_j \) and \( B_j, \ldots, B_k \) are transcripts of token games with at most \( 2^{\ell - 1} \) transfer steps.

Let \( u \) index a column which initially has \( a \) tokens in \( B_0 \) and ends with \( b \) tokens in \( B_k \), and let \( R \) be the set of tokens which end in column \( u \) but were not always in column \( u \). Clearly, \( |R| \geq b - a \). Let \( \{ t_1, \ldots, t_r \} \) be the tokens in \( R \) which are in the \( r \) highest positions in \( B_k \), and let \( R_0 = \{ t_1, \ldots, t_r \} \). Each token in \( R_0 \) has height at least \( b' \) in \( B_k \). Moreover, since the height of a token is non-increasing throughout the game, it follows that each token in \( R_0 \) has height at least \( b' \) in every array.

Since each \( t_i \in R_0 \) is moved to \( u \) during the token game, we may choose columns \( v_1, \ldots, v_r \) and indices \( \ell_1, \ldots, \ell_r \) such that \( t_i \) is moved from \( v_i \) in \( B_{\ell_i} \) to \( u \) in \( B_{\ell_i + 1} \). Since a token game forbids more than one transfer between a pair of columns, \( v_1, \ldots, v_r \) are distinct. Choose \( I \in \{ [0, j], [j, k] \} \) so that \( |R_1| \geq r/2 \), where \( R_1 = \{ t_i \in R_0 : \{ \ell_i, \ell_i + 1 \} \subseteq I \} \). Since \( t_i \) has height at least \( b' \) throughout the game, column \( v_i \) in \( B_{\ell_i} \) has at least \( b' \) grounded tokens.

Note that it is not possible for each of the columns in \( \{ v_i : t_i \in R_1 \} \) to begin the subgame \( \{ B_{\ell_i} : i \in I \} \) with more than \( a' \) tokens, since \( (a' + 1)|R_1| \geq (a' + 1)(r/2) > m \). It follows that some column \( v_i \) has at most \( a' \) tokens in the first array of \( \{ B_{\ell_i} : i \in I \} \) but has at least \( b' \) tokens in \( B_{\ell_i} \).

Iterating Lemma 3.4 gives the following.

**Lemma 3.5.** In a token game with \( m \) tokens and at most \( 2^\ell \) transfer steps, each column gains a net of at most \( 1 + 2\ell\sqrt{2m} \) tokens.

**Proof.** If \( \ell = 0 \), then the lemma is clear. For larger \( \ell \), suppose that there is a token game \( B_0, \ldots, B_k \) with at most \( 2^\ell \) transfer steps in which some column begins with \( a \) tokens and ends with \( b \) tokens.

We iterate Lemma 3.4 with \( a' = r = \sqrt{2m} \) to obtain, for each \( 1 \leq t \leq \ell \), a subgame with \( m \) tokens and at most \( 2^{\ell - t} \) transfer steps in which some column begins with at most \( a' \) tokens and ends with at least \( (b - a) - (2t - 1)r \) tokens.
With $\ell = t$, we obtain a token game with at most 1 transfer step in which some column begins with at most $a'$ tokens and ends with at least $(b - a) - (2\ell - 1)r$ tokens. We conclude $(b - a) - (2\ell - 1)r - a' \leq 1$, which implies $b - a \leq 1 + 2\ell r$.

**Corollary 3.6.** Always $\hat{g}(n, s) \leq 4\log n(\sqrt{2ns} + 1) + s + 1 = O(s + \sqrt{n}s \log n)$. In particular, if $n \geq \max\{2, s\}$, then $\hat{g}(n, s) \leq 11\sqrt{n}s \log n$.

**Proof.** Consider an $(n, s)$-token game in which some column starts with at most $s$ tokens and ends with $\hat{g}(n, s)$ tokens. Let $b = \hat{g}(n, s)$. Note that an $(n, s)$-token game contains at most $2\ell$ transfer steps provided that $2^\ell \geq \binom{n}{s}$; it suffices to choose $\ell = [2\log n]$. Let $m$ be the number of tokens in our $(n, s)$-token game; clearly $m \leq ns$. It now follows from Lemma 3.5 that $b - s \leq 1 + 2\ell [\sqrt{2m}] \leq 1 + 4\log n(\sqrt{2ns} + 1)$. When $n \geq \max\{2, s\}$, algebra gives the simpler bound $\hat{g}(n, s) \leq 11\sqrt{n}s \log n$.

Improvements to Corollary 3.6 directly translate to improved bounds on $f(G)$ via Lemma 2.5 and Lemma 3.3. Unfortunately, our next theorem shows that there is not much room to improve Corollary 3.6.

**Theorem 3.7.** Always $\hat{g}(n, s) \geq \max\{s, \sqrt{2ns} - 3s/2\}$. Consequently, $\hat{g}(n, s) \geq \Omega(s + \sqrt{n}s)$.

**Proof.** Clearly, $\hat{g}(n, s) \geq s$. Let $k$ be the largest integer such that $s^{(k + 1)/2} \leq n$ and note $k \geq [\sqrt{2n}/s - 1/2]$. Using that $n \geq s(1 + 2 + \cdots + k)$, we let $M_1, \ldots, M_k$ be disjoint sets of columns such that $|M_j| = sj$ for each $j$. Let $u_{j,1}, \ldots, u_{j,sj}$ be the columns in $M_j$. For each $j$, we construct a triangular pattern of tokens in $M_j$ so that for $1 \leq i \leq sj$, the column $u_{j,i}$ contains $i$ grounded tokens. We assume that the initial positions of all tokens are sufficiently high so that they fall into place as needed.

For $M_1$, we initialize the board so that for $1 \leq i \leq s$, the column $u_{1,i}$ starts with $i$ tokens. For $j \geq 2$, we assume that we have played the token game so that for $1 \leq i \leq s(j - 1)$, the column $u_{j-1,i}$ in $M_{j-1}$ contains $i$ grounded tokens. We use the tokens in $M_{j-1}$ to construct the desired pattern in $M_j$. We move the highest grounded token from each column in $u_{j-1,1}, \ldots, u_{j-1,s(j-1)}$ to $u_{j,sj}$ in order. Since $M_{j-1}$ has $s(j - 1)$ columns, this puts $s(j - 1)$ tokens in $u_{j,sj}$ and leaves a smaller triangular pattern in $M_{j-1}$ where $u_{j-1,i}$ contains $i$ grounded tokens. Next, we move the highest grounded token from each column in $u_{j-2,1}, \ldots, u_{j-1,s(j-1)}$ to $u_{j,sj-1}$ in order; this places $s(j - 1) - 1$ tokens in $u_{j,sj-1}$. Iterating this play, we move all tokens in $M_{j-1}$ to $M_j$. We complete the triangular pattern by allowing $\min\{i, s\}$ tokens whose initial positions were high in column $u_{j,i}$ to fall into place.

Note that we require at most $s$ tokens in each column initially. Moreover, since $M_1, \ldots, M_k$ are pairwise disjoint, no pair of columns is involved in more than one transfer step. After all steps, column $u_{k,sk}$ in $M_k$ contains $sk$ tokens, implying that $\hat{g}(n, s) \geq sk \geq s(\sqrt{2n}/s - 3/2)$.

Although the bounds in Corollary 3.6 and Theorem 3.7 establish the order of growth of $\hat{g}(n, s)$ up to a logarithmic factor, it would still be interesting to obtain the exact order of growth. If $\hat{g}(n, s) = O(s + \sqrt{n}s)$ as we suspect, then the log term in the lower bound in Corollary 4.2 can be removed. We do not know how sharp the inequality in Lemma 3.3 is; there may be room to make more substantial improvements to our upper bound on $g(n, s)$.
4 Dense Graphs

**Theorem 4.1.** Let $G$ be an $n$-vertex graph, let $s = n^{1/3}(11 \log n)^{2/3}$, and suppose that $n$ is sufficiently large so that $s \leq n - 2$. If $G$ has average degree $d$ and $d > 2$, then $f(G) > \frac{d}{4s}(1 - \frac{2}{3})(1 - \frac{1}{s})(1 - \frac{4s^2}{d - 2})$.

**Proof.** Let $H$ be a total ordering of $G$. Since $H$ has $nd/2$ edges, it follows that some edge $x_0x_1$ has height at least $d/2$. With $s' = \lfloor s \rfloor$, we apply Lemma 2.5 to obtain a monotone path of length at least $s'\left\lceil \frac{d}{2} - 1 \right\rceil + 1$. Using Lemma 3.3, Corollary 3.6, and monotonicity of $\hat{g}(n, s)$, we have $g(n, s') \leq \hat{g}(n - s', s') \leq \hat{g}(n, s') \leq 11\sqrt{ns' \log n}$. We compute

$$s'\left(\frac{d/2 - 1}{(s'+1) + g(n, s')}\right) + 1 > s'\left(\frac{d/2 - 1}{(s'/2) + 11\sqrt{ns' \log n}}\right)$$

$$\geq (s - 1)\left(\frac{d/2 - 1}{(s'/2) + 11\sqrt{ns' \log n}}\right)$$

$$= (s - 1)\left(\frac{d/2 - 1}{(s'/2) + s^2}\right)$$

$$\geq (s - 1)\left(\frac{d/2 - 1}{2s^2 - 1}\right)$$

$$= \frac{d}{4s} \left(1 - \frac{2}{d}\right) \left(1 - \frac{1}{s}\right) \left(1 - \frac{4s^2}{d - 2}\right)$$

When the average degree $d$ grows faster than $s^2$, Theorem 4.1 improves Rödl’s result $f(G) \geq (1 - o(1))\sqrt{d}$. In terms of $n$, an $n$-vertex graph must have average degree at least $Cn^{2/3}(\log n)^{4/3}$ for some constant $C$ in order for Theorem 4.1 to offer an improvement.

**Corollary 4.2.** $f(K_n) \geq \left(\frac{1}{20} - o(1))\left(\frac{n}{\log n}\right)^{2/3}$

**Proof.** By Theorem 4.1, we have $f(K_n) \geq \frac{n - 1}{4n^{1/3}(11 \log n)^{2/3}}(1 - o(1)) \geq \frac{n^{2/3}}{20(\log n)^{2/3}}(1 - o(1))$. \hfill \qed

We make no attempt to optimize the constant $1/20$. Echoing remarks of Graham and Kleitman [6], we conjecture that our lower bound on $f(K_n)$ is not sharp and that the order of growth of $f(K_n)$ is closer to linear than to $(n/\log n)^{2/3}$.

**References**


