Monotone Paths in Dense Edge-Ordered Graphs

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Monotone paths

- Let $G$ be a graph whose edges are ordered according to a labeling $\varphi$. 

What is $f(K_n)$?
Monotone paths

Let $G$ be a graph whose edges are ordered according to a labeling $\varphi$. 

![Diagram of a graph with labeled edges]
Monotone paths

- Let $G$ be a graph whose edges are ordered according to a labeling $\varphi$.

- A monotone path traverses edges in increasing order.
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- The altitude of $G$, denoted $f(G)$, is the maximum integer $k$ such that every edge-ordering of $G$ has a monotone path of length $k$. 
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- The altitude of $G$, denoted $f(G)$, is the maximum integer $k$ such that every edge-ordering of $G$ has a monotone path of length $k$.

- [Chvátal–Komlós (1971)] What is $f(K_n)$?
Prior work

Theorem (Graham–Kleitman (1973))

\[ \sqrt{n - \frac{3}{4} - \frac{1}{2}} \leq f(K_n) \leq \frac{3n}{4} \]
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- Alspach–Heinrich–Graham (unpublished):
  \[ f(K_n) \leq \left( \frac{7}{12} + o(1) \right) n \]
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**Theorem (Calderbank–Chung–Sturtevant (1984))**
\[f(K_n) \leq \left(\frac{1}{2} + o(1)\right)n\]
Prior work II

- Roditty–Shoham–Yuster (2001): the max. altitude of a planar graph is in \{5, 6, 7, 8, 9\}.


If \( p(n) = \omega(\log n / \sqrt{n}) \), then \( f(G(n, p)) \geq (1 - o(1)) \sqrt{n} \) with probability tending to 1.
Prior work II

- Roditty–Shoham–Yuster (2001): the max. altitude of a planar graph is in \( \{5, 6, 7, 8, 9\} \).
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- \( f(Q_n) \geq n/\lg n \)
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- \( f(Q_n) \geq n / \lg n \)
- If \( p(n) = \omega \left( \log n / \sqrt{n} \right) \), then \( f(G(n, p)) \geq (1 - o(1)) \sqrt{n} \) with probability tending to 1.
Random edge-orderings

Theorem (Lavrov–Loh (2015+))

- With probability tending to 1, a random edge-labeling of $K_n$ has a monotone path of length $0.85n$.
- With probability at least $\frac{1}{e} - o(1)$, a random edge-labeling of $K_n$ has a Hamiltonian monotone path.

Conjecture (Lavrov–Loh)

With high probability, a random edge-labeling of $K_n$ has a Hamiltonian monotone path.
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Theorem (Rödl (1973))
If \( G \) has average degree \( d \), then \( f(G) \geq (1 - o(1))\sqrt{d} \).
Our result

**Theorem (Graham–Kleitman (1973))**

\[ f(K_n) \geq \sqrt{n - \frac{3}{4}} - \frac{1}{2} \]

**Theorem (Rödl (1973))**

*If G has average degree d, then \( f(G) \geq (1 - o(1))\sqrt{d} \).*

**Theorem**

*Let G be an n-vertex graph, and let \( s = Cn^{1/3}(\lg n)^{2/3} \). If G has average degree d, then*

\[ f(G) \geq \frac{d}{4s} \left( 1 - \frac{2}{d} \right) \left( 1 - \frac{1}{s} \right) \left( 1 - \frac{4s^2}{d - 2} \right). \]
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\[
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\]

**Corollary**

\[ f(K_n) \geq (\frac{1}{20} - o(1))(n/\lg n)^{2/3} \]
The height table

- Let $G$ be a graph with vertices $w_1, \ldots, w_n$. 
The height table

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```

w_1

w_2  w_1
  5
5

w_3

w_2  w_3
  7  13
11

w_4

w_2  w_3  w_4
  10  14  14
15

w_5

w_2  w_3  w_4  w_5
  5  7  13  10
15

w_6

w_1  w_2  w_3  w_4  w_5  w_6
  11  5  2  6  4  9
12

w_5

w_2  w_3  w_4  w_5  w_6
  5  7  10  15  4
14

w_3

w_2  w_3  w_4
  5  7  14
10

w_4

w_2  w_3  w_4
  5  7  14
10

w_5

w_2  w_3  w_4  w_5
  5  7  14  15
10

w_6

w_1  w_2  w_3  w_4  w_5  w_6
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12
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- Let $G$ be a graph with vertices $w_1, \ldots, w_n$.

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  - Fill in the cells row by row, from bottom to top.
  - Next entry in column $i$ is the edge incident to $w_i$ with largest label not already appearing in $A$.
  - The height of an edge $e$, denoted $h(e)$, is the index of the row containing $e$. For example, $h(w_1w_2) = 3$. 

The diagram shows a graph with vertices $w_1, w_2, \ldots, w_6$ and labeled edges.
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\[
\begin{array}{ccccccc}
& w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \\
\hline
w_1 & 1 & 7 & 14 & 15 & & \\
w_2 & 5 & 13 & 2 & & & \\
w_3 & 11 & 12 & & & & \\
w_4 & 6 & 9 & 4 & 1 & & \\
w_5 & & & 6 & 8 & & \\
w_6 & & 5 & & & & \\
\end{array}
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```plaintext

\[
\begin{array}{cccccc}
16 & 1 & 2 & 3 & 4 & 5 \\
1 & 7 & 5 & 6 & 2 & 11 \\
2 & 5 & 6 & 2 & 12 & 9 \\
3 & 15 & 6 & 11 & 11 & 10 \\
4 & 15 & 11 & 10 & 10 & 10 \\
5 & 9 & 12 & 9 & 9 & 11 \\
6 & 4 & 2 & 9 & 12 & 12 \\
\end{array}
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$\begin{array}{ccccccc}
16 & 24 & & & & & \\
\hline
w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \\
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![Graph with vertices and edges labeled with numbers]

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    13  23  35  46  56  62
w_1  w_2  w_3  w_4  w_5  w_6
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\begin{array}{c|c|c|c|c|c}
    & w_1 & w_2 & w_3 & w_4 & w_5 \\
\hline
w_1 & 13 & 23 & 34 & 41 & 51 \\
16 & 24 & 35 & 46 & 56 & 62 \\
\hline
w_6
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- Next entry in column $i$ is the edge incident to $w_i$ with largest label not already appearing in $A$. 

<table>
<thead>
<tr>
<th></th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$w_4$</th>
<th>$w_5$</th>
<th>$w_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>25</td>
<td>–</td>
<td>45</td>
<td>51</td>
<td>63</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>23</td>
<td>34</td>
<td>41</td>
<td>56</td>
<td>62</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>24</td>
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The height table

- Let $G$ be a graph with vertices $w_1, \ldots, w_n$.

- The height table $A$ has a column for each vertex in $G$.

- Fill in the cells row by row, from bottom to top.

- Next entry in column $i$ is the edge incident to $w_i$ with largest label not already appearing in $A$.

- The height of an edge $e$, denoted $h(e)$, is the index of the row containing $e$. For example, $h(w_1w_2) = 3$. 

\[ 
\begin{array}{cccccc} 
12 & 25 & - & 45 & 51 & 63 \\
13 & 23 & 34 & 41 & 56 & 62 \\
16 & 24 & 35 & 46 & 56 & 62 \\
\hline 
w_1 & w_2 & w_3 & w_4 & w_5 & w_6 
\end{array} \]
Monotone path extension

- Given $G$, construct the height table $A$. 
Monotone path extension

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Let $x_0x_1$ be a max-height edge in column $x_0$. Set $P = x_0x_1$. 
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![Monotone path extension](image)
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- Let $e'$ be the highest such edge joining $x_k$ to a vertex outside $\{x_1, \ldots, x_{k-1}\}$.
- Extend $P$ along $e'$.
- Iteratively extending gives $f(G) \geq \lfloor 1/2 + \sqrt{d} \rfloor$, matching Rödl's bound asymptotically.
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![](https://via.placeholder.com/150)
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The algorithm

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\[ G' \]

\[ x_0 \quad x_1 \quad \ldots \quad x_{s-1} \quad x_s \]

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- Let $G' = G - \{x_0, \ldots, x_{s-1}\}$.
- Recursively find a long mono. path in $G'$ extending $x_s x_{s+1}$.
The algorithm

Analysis:

- Extending to $x_0 \ldots x_{s+1}$ uses at most $\binom{s+1}{2}$ rows of $A$. 

"G"
The algorithm

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Lemma

If $G$ has average degree $d$, then $f(G) \geq s \left\lceil \frac{d/2 - 1}{\binom{s+1}{2} + g(n, s)} \right\rceil$. 
The \((n, s)\)-token game
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\[ g(n, s) \leq \hat{g}(n - s, s) \]
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**Lemma**

\[ \Omega(s + \sqrt{ns}) \leq \hat{g}(n, s) \leq O(s + \sqrt{ns} \log n) \]
Summary

Lemma

If $G$ has average degree $d$, then $f(G) \geq s \left[ \frac{d/2-1}{\binom{s+1}{2}+g(n,s)} \right]$. 

Question

Can the bound $g(n,s) \leq O(s + \sqrt{ns \log n})$ be improved?
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Theorem
Let G be an n-vertex graph, and let \( s = C n^{1/3} (\log n)^{2/3} \). If G has average degree d, then

\[
f(G) \geq \frac{d}{4s} \left(1 - \frac{2}{d}\right) \left(1 - \frac{1}{2}\right) \left(1 - \frac{4s^2}{d - 2}\right).
\]

In particular, \( f(K_n) \geq (\frac{1}{20} - o(1))(n/\log n)^{2/3} \).
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*If* $G$ *has average degree* $d$, *then* $f(G) \geq s \left\lceil \frac{d/2 - 1}{(s+1)/2 + g(n,s)} \right\rceil$.

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If $G$ has average degree $d$, then $f(G) \geq s \left[ \frac{d/2 - 1}{(s+1)/2 + g(n,s)} \right]$.

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Thank You.