

Math 283 – Supplemental Handout
Proofs of statements with Quantifiers

We start with the easy one:

Proofs involving the Existential quantifier

When a statement to be proven simply states existence, an example is sufficient. For example, to prove *there is a real-valued solution to the equation $2x^2 - x - 1 = 0$* , simply showing that $2(1)^2 - 1 - 1 = 0$ is sufficient. Or, in general terms, to prove $(\exists x)P(x)$, it is sufficient to find such an x .

Proofs involving the Universal quantifier

These ones aren't so simple, but this is what we've been doing so far. In general, to prove $(\forall x)Q(x)$, it is necessary that we select an arbitrary x (i.e., we just call it x) and prove that $Q(x)$ holds. A simple algebraic example of this is a proof from the Jan. 22 exercises:

Proof that for all real numbers a and b , $a^2 + b^2 \geq 2ab$: Let $a, b \in \mathbb{R}$. Then, $(a - b)^2 \geq 0 \Rightarrow a^2 - 2ab + b^2 \geq 0 \Rightarrow a^2 + b^2 \geq 2ab$.

The first statement of the proof, “Let $a, b \in \mathbb{R}$ ” is the arbitrary selection of two real numbers. If we'd selected two actual real numbers – like 3.8 and 12 – then we wouldn't be proving this statement for **all** real numbers. Proofs of statements using the universal quantifier are the ones that we'll be doing when we use the techniques discussed beforehand (direct, contrapositive, and contradiction).

Proofs involving the Unique Existential quantifier

Proofs of statements $(\exists!x)R(x)$ can become arduous, but usually follow the same format. We first prove existence (see above). Then, to show that the example we've found is unique, we assume that two such examples exist and prove that they are actually the same. Symbolically, we prove that $R(x) \wedge R(y) \Rightarrow x = y$. Here is an elementary example of such a proof:

Proof that there exists a unique solution to $3x - 1 = 2$: To prove existence, observe that $3(1) - 1 = 2$. Now, assume that $3x - 1 = 3y - 1$. Then, $3x = 3y$ and so $x = y$. Therefore, $x = y$. Therefore, the unique solution is 1.

Here are some to try on your own:

- (1) Prove that there exists a real number between $\sqrt{2}$ and $\sqrt{3}$.
- (2) Prove that for all $x \in \mathbb{R}$, $x^2 + 2x < 0 \Rightarrow x < 0$.
- (3) Prove that the polynomial $p(x) = x^3 + x - 1$ has exactly one real root.