NAME (print): ____________________________

Topology Ph.D. Entrance Exam, August 2011

Write a solution of each exercise on a separate page.

Solve EACH of the exercises 1-3

Ex. 1. Let X and Y be Hausdorff topological spaces and let $f\colon X \to Y$ be continuous. Answer YES or NO for each of the following questions. In case your answer is "NO" give a counterexample for the statement. In case your answer is "YES" give a short argument. (Answer: "standard theorem" is acceptable, when appropriate.)

(a) If A is a compact subset of X, then $f[A]$ is a compact subset of Y.

(b) If A is a closed subset of X, then $f[A]$ is a closed subset of Y.

(c) If B is a compact subset of Y, then $f^{-1}(B)$ is a compact subset of X.

(d) If B is a closed subset of Y, then $f^{-1}(B)$ is a closed subset of X.

Ex. 2. Let $\langle X, \mathcal{T}_1\rangle$ and $\langle Y, \mathcal{T}_2\rangle$ be topological spaces.

(a) Define the product topology on $Z = X \times Y$.

(b) Prove that $\mathrm{cl}(A) \times \mathrm{cl}(B) = \mathrm{cl}(A \times B)$ for every $A \subset X$ and $B \subset Y$.

Ex. 3. Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces. Prove that the following two definitions of continuity of $f\colon X \to Y$ are equivalent:

(a) (topological definition) $f^{-1}(U) \in \mathcal{T}$ for every $U \in \mathcal{T}$.

(b) (ε-δ definition) For every $x_0 \in X$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $x \in X$, if $d(x, x_0) < \delta$, then $\rho(f(x), f(x_0)) < \varepsilon$.

Solve TWO of the exercises 4-6

Ex. 4. Let X and Y be Hausdorff topological. Recall that a graph of a function $f: X \to Y$ is defined as $G(f) = \{\langle x, f(x) \rangle \in X \times Y : x \in X\}$ and that, for a metric space $\langle Z, d \rangle$ and non-empty sets $A, B \subset Z$, their distance is defined as $\text{dist}(A, B) = \inf\{d(a, b) : a \in A \ \& \ b \in B\}$.

(a) Show that if f is continuous, then its graph $G(f)$ is a closed subset of $X \times Y$.

(b) Show that if $f, g: [0, 1] \to [0, 1]$ are continuous functions, then

$$\text{dist}(G(f), G(g)) = 0 \text{ if, and only if, } f(x) = g(x) \text{ for some } x \in X.$$

Note: The interval $[0, 1]$ and its square $[0, 1]^2$ are considered with the standard Euclidean distance.

Ex. 5. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Show that $f[\mathbb{R}^2 \setminus S]$ is an interval (possibly improper) for every countable set $S \subset \mathbb{R}^2$.

Ex. 6. Recall, that a topological space is zero-dimensional provided it has a basis formed by clopen (i.e., simultaneously closed and open) sets. Show that every countable normal topological space X is zero-dimensional.

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Topology Ph.D. Entrance Exam, August 2012

Write a solution of each exercise on a separate page.

Ex. 1. Let $\langle X, \tau\rangle$ be a topological space and let $\{A_t\}_{t\in T}$ be an indexed family of arbitrary subsets of X. Determine each of the following statements by either proving it or providing a counterexample, where $\mathrm{cl}(A)$ stands for the closure of a set A.

(a) $\bigcap_{t\in T} \mathrm{cl}(A_t) \subset \mathrm{cl}\left(\bigcap_{t\in T} A_t\right)$

(b) $\mathrm{cl}\left(\bigcap_{t\in T} A_t\right) \subset \bigcap_{t\in T} \mathrm{cl}(A_t)$

Ex. 2. Show that a continuous image of a separable space is separable, that is, if there exists a continuous function from a separable topological space X *onto* a topological space Y, then Y is separable. Include the definition of a separable topological space.

Ex. 3. Let f be a continuous function from a compact Hausdorff topological space X into a Hausdorff topological space Y. Consider $X \times Y$ with the product topology. Show that the map $h\colon X \to X \times Y$ given by the formula $h(x) = \langle x, f(x)\rangle$ is a homeomorphic embedding.

Ex. 4. For the topologies τ and σ on $\mathbb{R}$ let symbol $C(\tau, \sigma)$ stand for the family of all continuous functions from $\langle \mathbb{R}, \tau\rangle$ into $\langle \mathbb{R}, \sigma\rangle$.

Let $\mathcal{T}_s$ be the standard topology on $\mathbb{R}$ and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on $\mathbb{R}$ such that $C(\mathcal{T}_1, \mathcal{T}_2) = C(\mathcal{T}_s, \mathcal{T}_s)$. Show that:

(i) $\mathcal{T}_2 \subseteq \mathcal{T}_1$, that is, $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

(ii) $\mathcal{T}_2 \neq \{\emptyset, \mathbb{R}\}$, that is, $\mathcal{T}_2$ is not trivial.

(iii) $\langle \mathbb{R}, \mathcal{T}_1\rangle$ is connected.

(Notice that $C(\mathcal{T}_1, \mathcal{T}_2) = C(\mathcal{T}_s, \mathcal{T}_s)$ does not imply that either of the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ must be equal to the standard topology $\mathcal{T}_s$.)

Ex. 5. Consider the following subsets, $\vdash$ and $\models$, of $\mathbb{R}^2$, where $\mathbb{R}^2$ is endowed with the standard topology:

$$\vdash = (\{0\} \times [-2, 2]) \cup ([0, 2] \times \{0\}) \quad \& \quad \models = (\{0\} \times [-2, 2]) \cup ([0, 2] \times \{-1, 1\}).$$

Prove, or disprove the following:

(i) There exists a continuous function from $\vdash$ onto $\models$.

(ii) There exists a continuous function from $\models$ onto $\vdash$.

Your argument must be precise, but no great details are necessary.

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Topology Ph.D. Entrance Exam, August 2013

Solve the following five exercises. Write a solution of each exercise on a separate page. In what follows the symbols $\mathrm{int}(A)$, $\mathrm{cl}(A)$, and A' stand for the interior, closure, and the set of limit points of A, respectively.

Ex. 1. Prove, or disprove by an example, that each of the following properties holds for every subset A of a topological space X.

(a) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

(b) $(A')' = A'$.

Ex. 2. A topological space is a T_0-space provided for every distinct $x, y \in X$ there exists an open set U in X which contains precisely one of the points x and y. Show that X is a T_0-space if, and only if, $\mathrm{cl}(\{x\}) \neq \mathrm{cl}(\{y\})$ for all distinct $x, y \in X$.

Ex. 3. Let $\{A_s \colon s \in \mathbb{R}\}$ be a family of connected subsets of a topological space X. Assume that $A_s \cap A_t \neq \emptyset$ for every $s, t \in \mathbb{R}$. Show that $A = \bigcup_{s\in\mathbb{R}} A_s$ is connected. (Note, that we do *not* assume that $\bigcap_{s\in\mathbb{R}} A_s \neq \emptyset$.)

Ex. 4. Let $\langle X, d\rangle$ be a metric space and let $A \subset X$ be such that it has no limit points in X, that is, such that $A' = \emptyset$. Show that there exists a family $\{U_a\}_{a\in A}$ of pairwise disjoint open sets such that $a \in U_a$ for every $a \in A$.

Ex. 5. Let X be completely regular; let A and B be disjoint closed subsets of X. Show that if A is compact, there is a continuous function $f \colon X \to [0, 1]$ such that $f[A] \subset \{0\}$ and $f[B] \subset \{1\}$. (Note, that we do *not* assume that X is normal.)

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Topology Ph.D. Entrance Exam, May 2015

Solve the following five exercises. Write a solution of each exercise on a separate page. In what follows the symbol int(A) stands for the interior of A. Any subset of $\mathbb{R}$ is considered with the standard topology.

Ex. 1. Let $\langle A_i \rangle_{i=1}^{\infty}$ be an arbitrary sequence of subsets of a topological space X. Show that for any natural number k we have

$$\operatorname{int}\left(\bigcap_{i=1}^{\infty} A_i\right) = \left(\bigcap_{i=1}^{k} \operatorname{int}(A_1)\right) \cap \operatorname{int}\left(\bigcap_{i=k+1}^{\infty} A_i\right).$$

Ex. 2. Let X be a Hausdorff topological space. Show that for every compact subset B of X and any $a \in X \setminus B$ there exist disjoint sets U and V open in X such that $a \in U$ and $B \subset V$. *Do not assume that X is regular!*

Ex. 3. Prove or give a counterexample: The product of two path-connected spaces is also path-connected.

Ex. 4. Let X be an arbitrary topological space and let $\mathbb{Z}$ stand for the set of all integers. Let $\{A_k : k \in \mathbb{Z}\}$ be a family of connected subsets of X. Show that if $A_k \cap A_{k+1} \neq \emptyset$ for every $k \in \mathbb{Z}$, then $\bigcup_{k \in \mathbb{Z}} A_k$ is a connected subset of X.

Ex. 5. Let X be a compact topological space and let $f : X \to \mathbb{R}$ be an arbitrary, **not necessary continuous**, function. Assume that f is locally bounded, that is, that for every $x \in X$ there exists an open $U \ni x$ such that $f[U]$ is bounded in $\mathbb{R}$. Show that $f[X]$ is bounded in $\mathbb{R}$.

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Topology Ph.D. Entrance Exam, April 2016

Solve the following five exercises. Write a solution of each exercise on a separate page. In what follows the symbols $\text{int}(A)$ and $\text{cl}(A)$ stand, respectively, for the interior and the closure of A. Any subset of $\mathbb{R}$ is considered with the standard topology, unless stated otherwise.

Ex. 1. Prove, or disprove by giving a counterexample, each of the following statements.

(i) The product of two regular spaces is a regular space.

(ii) The product of two normal spaces is a normal space.

Ex. 2. Prove, directly from the definition, that a compact Hausdorff space is regular. Include the definitions of Hausdorff and regular regular topological spaces.

Ex. 3. Prove that $[0, 1]$, considered with the standard topology, is compact. You can use, without a proof, the standard facts on the order of $\mathbb{R}$.

Ex. 4. Is a continuous image of a separable space separable? Prove it, or give a counterexample. Include a definition of separable topological space.

Ex. 5. Consider $\mathbb{R}^n$, $n \geq 1$, with the standard metric.

(i) Show that an open subset U of $\mathbb{R}^n$ is connected if, and only if, it is path connected. **Hint.** Fix an $x \in U$ and show that the following set $\{y \in U\colon \text{ there is a path in } U \text{ from } x \text{ to } y\}$ is both closed and open in U.

(ii) Give an example of a closed subset F of $\mathbb{R}^n$ which is connected but not path connected.