Math. 156
Instr. K. Ciesielski

Sample Test # 2.

Ex. 1. Evaluate the limits

(a) $\lim_{t\to\infty} \dfrac{t^2+1}{t\ln t} =$

(b) $\lim_{x\to 0} \dfrac{\ln(1+x^2)}{e^x - \cos x} =$

(c) $\lim_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{\ln(1+x)} \right) =$

(d) $\lim_{x\to 1^+} (x-1)^{\ln x} =$

Ex. 2. Evaluate the integrals

(a) $\displaystyle\int \frac{1}{9+4x^2}\, dx =$

(b) $\displaystyle\int \frac{\sec^2\theta}{1+\tan\theta}\, d\theta =$

(c) $\displaystyle\int \cos^2 5y\, dy =$

(d) $\displaystyle\int \sin^3 t\ \cos^3 t\, dt =$

(e) $\displaystyle\int \tan x\ \sec^4 x\, dx =$

(f) $\displaystyle\int x \arctan x\, dx =$

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Solutions to Sample Test # 2.

Ex. 1. (a)

$$\lim_{t\to\infty} \frac{t^2+1}{t\ln t} = \left(\frac{\infty}{\infty};\ \text{L'Hôpital's Rule}\right)$$

$$\lim_{t\to\infty} \frac{2t}{\ln t + t\,\frac{1}{t}} = \lim_{t\to\infty} \frac{2t}{\ln t + 1} = \left(\frac{\infty}{\infty};\ \text{L'Hôpital's Rule}\right)$$

$$\lim_{t\to\infty} \frac{2}{1/t} = \lim_{t\to\infty} 2t = \infty$$

(b)

$$\lim_{x\to 0} \frac{\ln(1+x^2)}{e^x - \cos x} = \left(\frac{\ln 1}{e^0 - \cos 0} = \frac{0}{0};\ \text{L'Hôpital's Rule}\right)$$

$$\lim_{x\to 0} \frac{\frac{1}{1+x^2}\,2x}{e^x + \sin x} = \frac{\frac{1}{1+0}\,0}{e^0 + 0} = \frac{0}{1} = 0$$

(c)

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\ln(1+x)}\right) = \lim_{x\to 0} \frac{\ln(1+x) - x}{x\ln(1+x)} = \left(\frac{\ln 1}{0 \ln 1} = \frac{0}{0};\ \text{L'Hôpital's Rule}\right)$$

$$\lim_{x\to 0} \frac{1/(1+x) - 1}{\ln(1+x) + x/(1+x)} = \lim_{x\to 0} \frac{\frac{1-(1+x)}{1+x}}{\frac{(1+x)\ln(1+x) + x}{1+x}} =$$

$$\lim_{x\to 0} \frac{-x}{(1+x)\ln(1+x) + x} = \left(\frac{0}{0};\ \text{L'Hôpital's Rule}\right)$$

$$\lim_{x\to 0} \frac{-1}{[\ln(1+x) + (1+x)/(1+x)] + 1} = \frac{-1}{[\ln 1 + 1] + 1} = -\frac{1}{2}$$

(d)

$$\lim_{x\to 1^+} (x-1)^{\ln x} = \exp\left(\lim_{x\to 1^+} \ln[(x-1)^{\ln x}]\right) = \exp\left(\lim_{x\to 1^+} (\ln x)\ln(x-1)\right) =$$

$$\exp\left(\lim_{x\to 1^+} \frac{\ln(x-1)}{[\ln x]^{-1}}\right) = \left(\frac{-\infty}{\infty};\ \text{L'Hôpital's Rule}\right)$$

$$\exp\left(\lim_{x\to 1^+} \frac{1/(x-1)}{-[\ln x]^{-2}/x}\right) = \exp\left(\lim_{x\to 1^+} -\frac{\frac{1}{x-1}}{\frac{1}{x\,[\ln x]^2}}\right) =$$

$$\exp\left(\lim_{x\to 1^+} \frac{x\,[\ln x]^2}{x-1}\right) = \left(\frac{0}{0};\ \text{L'Hôpital's Rule}\right)$$

$$\exp\left(\lim_{x\to 1^+} \frac{[\ln x]^2 + x\,[2(\ln x)/x]}{1}\right) = \exp([\ln 1]^2 + 0) = e^0 = 1.$$

Ex. 2. (a) $\int \frac{1}{9+4x^2}\,dx = \int \frac{1}{9\left(1+\frac{4}{9}x^2\right)}\,dx = \int \frac{1}{9}\,\frac{1}{1+\left(\frac{2}{3}x\right)^2}\,dx$

Put $u = \frac{2}{3}x$; then $\frac{du}{dx} = \frac{2}{3}$ and $dx = \frac{3}{2}du$. So

$$\int \frac{1}{9+4x^2}\,dx = \int \frac{1}{9}\,\frac{1}{1+u^2}\,\frac{3}{2}du = \frac{1}{6}\arctan u + C = \frac{1}{6}\arctan\left(\frac{2}{3}x\right) + C$$

(b) $\int \frac{\sec^2\theta}{1+\tan\theta}\,d\theta$

Put $u = 1 + \tan\theta$; then $\frac{du}{d\theta} = \sec^2\theta$ and $du = \sec^2\theta\,d\theta$. So

$$\int \frac{\sec^2\theta}{1+\tan\theta}\,d\theta = \int \frac{1}{u}\,du = \ln|u| + C = \ln|1+\tan\theta| + C$$

(c) $\int \cos^2 5y\,dy = \int \frac{1}{2}(1+\cos 10y)\,dy$

Put $u = 10y$; then $\frac{du}{dy} = 10$. So $dy = \frac{du}{10}$ and

$$\int \cos^2 5y\,dy = \int \frac{1}{2}(1+\cos u)\,\frac{du}{10} = \frac{1}{20}(u+\sin u) + C = \frac{1}{20}(10y + \sin 10y) + C$$

(d) $\int \sin^3 t\,\cos^3 t\,dt = \int \sin^3 t\,\cos^2 t\,\cos t\,dt = \int \sin^3 t\,(1-\sin^2 t)\cos t\,dt$

Put $u = \sin t$; then $\frac{du}{dt} = \cos t$. So $du = \cos t\,dt$ and

$$\int \sin^3 t\cos^3 t\,dt = \int u^3(1-u^2)\,du = \int u^3 - u^5\,du = \frac{1}{4}u^4 - \frac{1}{6}u^6 + C = \frac{1}{4}\sin^4 t - \frac{1}{6}\sin^6 t + C$$

(e) $\int \tan x \ \sec^4 x \, dx = \int \tan x \ \sec^2 x \sec^2 x \, dx = \int \tan x \, (1 + \tan^2 x) \sec^2 x \, dx$

Put $u = \tan x$; then $\dfrac{du}{dx} = \sec^2 x$. So $du = \sec^2 x \, dx$ and

$$\int \tan x \sec^4 x \, dx = \int u(1 + u^2) \, du = \int u + u^3 \, du = \frac{1}{2}u^2 + \frac{1}{4}u^4 + C = \frac{1}{2} \tan^2 x + \frac{1}{4}\tan^4 x + C$$

(f) $\int x \arctan x \, dx$

By parts. Consider $u = \arctan x$ and $dv = x$. Then

$$\int x \arctan x \, dx = \int u \, dv = uv - \int v \, du = (\arctan x) \frac{1}{2}x^2 - \int \frac{1}{1 + x^2} \, \frac{1}{2}x^2 \, dx =$$

$$\frac{1}{2}x^2 \arctan x - \int \frac{1}{2} \frac{x^2}{1 + x^2} \, dx = \frac{1}{2}x^2 \arctan x - \int \frac{1}{2} \frac{(1 + x^2) - 1}{1 + x^2} \, dx =$$

$$\frac{1}{2}x^2 \arctan x - \int \frac{1}{2}\left(1 - \frac{1}{1 + x^2}\right) dx = \frac{1}{2}x^2 \arctan x - \frac{1}{2}(x - \arctan x) + C =$$

$$\frac{1}{2}(x^2 + 1)\arctan x - \frac{1}{2}x + C$$