SAMPLE TEST # 4

Solve the following exercises. **Show your work.** (No credit will be given for an answer with no supporting work shown.)

**Ex. 1.** Transform the following system of equations into a single second order equation in terms of $x_1$. Then give the initial condition for the resulted equation that corresponds to the given initial conditions. Do not solve.

$x_1' = -0.5x_1 + 2x_2; \quad x_2' = -2x_1 - 0.5x_2; \quad x_1(0) = -2, \quad x_2(0) = 2.$

**Solution:** From the first equation we get $x_2 = 0.5x_1' + 0.25x_1$. Substituting this to the second equation gives $(0.5x_1' + 0.25x_1)' = -2x_1 - 0.5(0.5x_1' + 0.25x_1)$, which in turn leads to $0.5x_1'' + 0.25x_1' = -2x_1 - 0.25x_1' - 0.125x_1$. Multiplying this equation by 8 produces $4x_1'' + 2x_1' = -16x_1 - 2x_1' - x_1$, so $4x_1'' = -4x_1' - 17x_1$.

Clearly $x_1(0) = -2$. To calculate $x_1'(0)$ we put $t = 0$ to the first equation and use given boundary values: $x_1'(0) = -0.5x_1(0) + 2x_2(0) = -0.5(-2) + 2 \cdot 2 = 5$.

Answer: $4x_1'' = -4x_1' - 17x_1; \quad x_1(0) = -2; \quad x_1'(0) = 5$.

**Ex. 2.** Use eigenvalues and eigenvectors to find the general solution of the given systems of differential equations. The solution must be expressed in terms of real-valued functions.

(a) $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$

**Solution:** The eigenvalues are obtained as roots of the equation

$\det\left( \begin{array}{cc} 1-r & -2 \\ 3 & -4-r \end{array} \right) = 0$, that is, $(1-r)(-4-r) + 6 = 0$, or $r^2 + 3r - 4 + 6$. Hence, $(r + 2)(r + 1) = 0$, leading to the eigenvalues $-1$ and $-2$.

The eigenvalue $r = -1$ leads to the eigenvector equation

$\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

so $2\xi_1 - 2\xi_2 = 0$. Thus, $\xi_2 = \xi_1$, $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and the eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The eigenvalue $r = -2$ leads to the eigenvector equation

$\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

so $3\xi_1 - 2\xi_2 = 0$. Thus, $\xi_2 = 1.5\xi_1$, $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ 1.5\xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 1.5 \end{pmatrix}$, and the eigenvector is $\begin{pmatrix} 1 \\ 1.5 \end{pmatrix}$.

Answer: $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1.5 \end{pmatrix} e^{-2t}$. 

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(b) $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \mathbf{x}$

**Solution:** The eigenvalues are obtained as roots of the equation
\[
\det \begin{pmatrix} 1 - r & 2 \\ -5 & -1 - r \end{pmatrix} = 0,
\]
that is, $(1 - r)(-1 - r) + 10 = 0$, or $r^2 - 1 + 10 = 0$. Hence, we have two complex the eigenvalues: $\pm 3i$.

Eigenvalue $r = 3i$ leads to eigenvector equation
\[
\begin{pmatrix} 1 - 3i & 2 \\ -5 & -1 - 3i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]
so $(1 - 3i)\xi_1 + 2\xi_2 = 0$. Thus, $\xi_2 = (1.5i - 0.5)\xi_1$,
\[
\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ (1.5i - 0.5)\xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 1.5i - 0.5 \end{pmatrix},
\]
and the eigenvector is $\begin{pmatrix} 1 \\ 1.5i - 0.5 \end{pmatrix}$. Since the real part of the eigenvalue is 0, this leads to the complex solution $x^{(1)}(t) = \begin{pmatrix} 1 \\ 1.5i - 0.5 \end{pmatrix} e^{0t}(\cos 3t + i \sin 3t)$. Therefore
\[
x^{(1)}(t) = \begin{pmatrix} \cos 3t + i \sin 3t \\ -0.5 \cos 3t - 1.5 \sin 3t + i(1.5 \cos 3t - 0.5 \sin 3t) \end{pmatrix}
\]
and
\[
x^{(1)}(t) = \begin{pmatrix} \cos 3t \\ -0.5 \cos 3t - 1.5 \sin 3t \end{pmatrix} + i \begin{pmatrix} \sin 3t \\ 1.5 \cos 3t - 0.5 \sin 3t \end{pmatrix}.
\]
Answer: $x(t) = c_1 \begin{pmatrix} \cos 3t \\ -0.5 \cos 3t - 1.5 \sin 3t \end{pmatrix} + c_2 \begin{pmatrix} \sin 3t \\ 1.5 \cos 3t - 0.5 \sin 3t \end{pmatrix}$.

(c) $\mathbf{x}' = \begin{pmatrix} 6 & -3 \\ 3 & 0 \end{pmatrix} \mathbf{x}$

**Solution:** The eigenvalues are obtained as roots of the equation
\[
\det \begin{pmatrix} 6 - r & -3 \\ 3 & -r \end{pmatrix} = 0,
\]
that is, $(6 - r)(-r) + 9 = 0$, or $(r - 3)^2 = 0$. Hence, we have one double eigenvalue: $r = 3$.

The first eigenvector equation is
\[
\begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]
so $3\xi_1 - 3\xi_2 = 0$. Thus, $\xi_2 = \xi_1$, \[\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix},\]
and the eigenvector is $\xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This gives the first fundamental solution $x^{(1)} = \xi e^{3t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$. Recall that the second fundamental solution is of the form $x^{(2)} = \xi t e^{3t} + \eta e^{3t}$, where $\eta$ is one of the solutions of the system
\[
\begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \xi,
\]
that is, \[\begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]
Hence, $3\eta_1 - 3\eta_2 = 1$, that is, $\eta_1 = \eta_2 + 1/3$. So, $\eta = \begin{pmatrix} \eta_2 + 1/3 \\ \eta_2 \end{pmatrix} = \eta_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$. One of these solutions is when $\eta_2 = 0$, giving $\eta = \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$. 

Solution: The general solution of the heat equation

Solve the heat equation:

Ex. 5. \[ x \text{equations} \]

be used to our partial differential equation and it leads to the pair of ordinary differential equations. If \( \lambda \) is equal to a constant, which we denote by \( \lambda \), where either of the two formats is acceptable as an answer.

This leads us to a final solution

\[
y \left( 0 \right) = 0 \quad \text{and} \quad y \left( \pi \right) = 0.
\]

Thus, the general solution of the equation is of the form

\[
y = c_1 \cos \left( \frac{\pi x}{L} \right) \exp \left( -\frac{\pi^2 \alpha^2}{L^2} \right) t + c_2 \sin \left( \frac{\pi x}{L} \right) \exp \left( -\frac{\pi^2 \alpha^2}{L^2} \right) t.
\]

Thus, the boundary value problem has infinitely many solutions, each of the form \( y(t) = c \sin 2t \), where \( c \) is an arbitrary constant.

Ex. 4. Determine whether the method of separation of variables can be used to replace the partial differential equation \( u_{xx} + u_{xt} + u_t = 0 \) by a pair of ordinary differential equations. If so, find the ordinary differential equations. Do not solve them.

Solution: We assume that \( u \) is of the form \( u(x, t) = X(x)T(t) \). Then we have \( u_{xx} = X''T, u_{xt} = X'T', \) and \( u_t = XT' \). Substituting this to the equation gives \( X''T + X'T' + XT' = 0 \). So, \( X'' + (X' + X)' = 0 \) and \( X'' = -(X' + X)' \). Thus, \( \frac{X''}{X' + X} = -\frac{T'}{T} \) and this quantity is equal to a constant, which we denote by \( \lambda \). Thus the method of separation of variables can be used to our partial differential equation and it leads to the pair of ordinary differential equations \( X'' = \lambda (X' + X) \) and \( T' = -\lambda T \).

Ex. 5. Solve the heat equation: \( u_t = 9u_{xx}, u(0, t) = u(2, t) = 0, u(x, 0) = 13 \) for \( 0 < x < 2 \).

Solution: The general solution of the heat equation

\[
u_t = \alpha^2 u_{xx} \quad u(0, t) = u(L, t) = 0 \quad u(x, 0) = f(x) \quad 0 < x < L
\]

is given by 

\[
u(x, t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2 \pi^2 \alpha^2}{L^2} t} \sin \frac{n \pi x}{L}, \quad \text{where} \quad c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n \pi x}{L} dx.
\]

In our case \( L = 2 \) and \( \alpha^2 = 9 \), reducing the solution to

\[
u(x, t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2 (9 \pi^2 / 4)}{L^2} t} \sin \frac{n \pi x}{2}, \quad \text{where}
\]

\[
c_n = \frac{2}{2} \int_0^2 u(x, 0) \sin \frac{n \pi x}{2} dx = \int_0^2 13 \sin \frac{n \pi x}{2} dx
\]

\[
= -13 \frac{2}{n \pi} \cos \frac{n \pi x}{2} \bigg|_0^2
\]

\[
= -13 \frac{2}{n \pi} \left( \cos n \pi - \cos 0 \right)
\]

\[
= \frac{26}{n \pi} \left( \cos \pi - \cos 0 \right) = \left\{ \begin{array}{ll} 0 & \text{if} \ n \text{ is even} \\ \frac{52}{n \pi} & \text{if} \ n \text{ is odd} \end{array} \right.
\]

This leads us to a final solution

\[
u(x, t) = \sum_{n=1, 3, 5, \ldots}^{\infty} \frac{52}{n \pi} e^{-\frac{n^2 (9 \pi^2 / 4)}{L^2} t} \sin \frac{n \pi x}{2} = \sum_{k=0}^{\infty} \frac{52}{(2k+1) \pi} e^{-\frac{(2k+1)^2 (9 \pi^2 / 4)}{L^2} t} \sin \frac{(2k+1) \pi x}{2},
\]

where either of the two formats is acceptable as an answer.