Ex. 2.24. Define \( f: \mathbb{R} \to \mathbb{R} \) by \( f(x) = x^2 \). Show that \( f \) is continuous.

**Proof.** Let \( p \in \mathbb{R} \) and \( A \subset \mathbb{R} \) be such that \( p \sim A \). We need to show that \( p^2 = f(p) \sim f[A] \). (Note that \( p \) need not to belong to \( A \)!) In the proof we will use the characterization from Thm. 2.3 that for \( p \subset A \) and so \( p < f \) and so \( f \) will use the characterization from Thm. 2.3 that for \( z \in \mathbb{R} \) and \( B \subset \mathbb{R} \)

\((*) z \sim B \) if and only if \( I \cap B \neq \emptyset \) for every open interval \( I \ni z \).

So, let \( I \ni f(p) = p^2 \) be an open interval. We need to show that \( I \cap f[A] \neq \emptyset \).

**Case** \( p > 0 \). Since \( p^2 > 0 \) we can find \( a, b > 0 \) such that \( p^2 \in (a, b) \subset I \). Then \( 0 < a < p^2 < b \). So, \( \sqrt{a} < p < \sqrt{b} \), as \( p > 0 \). In particular, \( p \in J = (\sqrt{a}, \sqrt{b}) \).

Since \( p \sim A \), by \((*)\) we have \( J \cap A \neq \emptyset \). Thus, there exists an \( x \in J \cap A \).

In particular, \( x \in J \), that is, \( \sqrt{a} < x < \sqrt{b} \). Therefore, \( a < x^2 < b \) and so \( f(x) = x^2 \in (a, b) \subset I \). Moreover, \( f(x) \in f[A] \), since \( x \in A \). Thus, \( f(x) \in I \cap f[A] \), and so \( I \cap f[A] \neq \emptyset \). This finishes the case \( p > 0 \).

**Case** \( p < 0 \). Since \( p^2 > 0 \) we can find \( a, b > 0 \) such that \( p^2 \in (a, b) \subset I \). Then \( 0 < a < p^2 < b \). So, \( -\sqrt{b} < p < -\sqrt{a} \), as \( p < 0 \). In particular, \( p \in J = (-\sqrt{b}, -\sqrt{a}) \).

Since \( p \sim A \), by \((*)\) we have \( J \cap A \neq \emptyset \). Thus, there exists an \( x \in J \cap A \).

In particular, \( x \in J \), that is, \( -\sqrt{a} < x < -\sqrt{b} \). Therefore, \( a < x^2 < b \) and so \( f(x) = x^2 \in (a, b) \subset I \). Moreover, \( f(x) \in f[A] \), since \( x \in A \). Thus, \( f(x) \in I \cap f[A] \), and so \( I \cap f[A] \neq \emptyset \). This finishes the case \( p < 0 \).

**Case** \( p = 0 \). Since \( 0 = f(0) \in I \) we can find an \( \varepsilon > 0 \) such that \((-\varepsilon, \varepsilon) \subset I \). Let \( J = (-\sqrt{\varepsilon}, \sqrt{\varepsilon}) \ni 0 \).

Since \( 0 \sim A \), by \((*)\) we have \( J \cap A \neq \emptyset \). Thus, there exists an \( x \in J \cap A \).

In particular, \( x \in J \), that is, \(|x| < \sqrt{\varepsilon} \). Therefore, \( x^2 < \varepsilon \) and so \( f(x) = x^2 \in (-\varepsilon, \varepsilon) \subset I \). Moreover, \( f(x) \in f[A] \), since \( x \in A \). Thus, \( f(x) \in I \cap f[A] \), and so \( I \cap f[A] \neq \emptyset \). This finishes the case \( p = 0 \).

We considered all possible cases for \( p \), so \( f \) is continuous at every \( p \in \mathbb{R} \).
Ex. 2.26. Define $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $f(x) = \frac{1}{x}$. Show that $f$ is continuous.

**Proof.** Let $p \in \mathbb{R} \setminus \{0\}$ and $A \subset \mathbb{R}$ be such that $p \sim A$. We need to show that $\frac{1}{p} = f(p) \sim f[A]$. (Note that $p$ need not to belong to $A$!) In the proof we will use the characterization from Thm. 2.3 that for $z \in \mathbb{R}$ and $B \subset \mathbb{R}$

\[ (*) \ z \sim B \ \text{if and only if} \ I \cap B \neq \emptyset \ \text{for every open interval} \ I \ni z. \]

So, let $I \ni f(p) = \frac{1}{p}$ be an open interval. We need to show that $I \cap f[A] \neq \emptyset$.

**Case $p > 0$.** Since $\frac{1}{p} > 0$ we can find $a, b > 0$ such that $\frac{1}{p} \in (a, b) \subset I$. Then $0 < a < \frac{1}{p} < b$. So, $\frac{1}{b} < p < \frac{1}{a}$. In particular, $p \in J = (\frac{1}{b}, \frac{1}{a})$.

Since $p \sim A$, by $(*)$ we have $J \cap A \neq \emptyset$. Thus, there exists an $x \in J \cap A$. In particular, $x \in J$, that is, $0 < \frac{1}{b} < p < \frac{1}{a}$. Therefore, $a < \frac{1}{x} < b$ and so $f(x) = \frac{1}{x} \in (a, b) \subset I$. Moreover, $f(x) \in f[A]$, since $x \in A$. Thus, $f(x) \in I \cap f[A]$, and so $I \cap f[A] \neq \emptyset$. This finishes the case $p > 0$.

**Case $p < 0$.** Since $\frac{1}{p} < 0$ we can find $a, b < 0$ such that $\frac{1}{p} \in (a, b) \subset I$. Then $a < \frac{1}{p} < b < 0$. So, $\frac{1}{b} < p < \frac{1}{a}$. In particular, $p \in J = (\frac{1}{b}, \frac{1}{a})$.

Since $p \sim A$, by $(*)$ we have $J \cap A \neq \emptyset$. Thus, there exists an $x \in J \cap A$. In particular, $x \in J$, that is, $\frac{1}{b} < p < \frac{1}{a} < 0$. Therefore, $a < \frac{1}{x} < b$ and so $f(x) = \frac{1}{x} \in (a, b) \subset I$. Moreover, $f(x) \in f[A]$, since $x \in A$. Thus, $f(x) \in I \cap f[A]$, and so $I \cap f[A] \neq \emptyset$. This finishes the case $p < 0$.

We considered all possible cases of $p \in \mathbb{R} \setminus \{0\}$, so $f$ is continuous.
Ex. 2.30. If $A, B \subseteq \mathbb{R}$ such that $A \subseteq B$ then $A^\ell \subseteq B^\ell$.

Proof. **Method I.** Let $x \in A^\ell$. Then, by definition, $x \sim A \setminus \{x\}$. So, by Thm. 2.3, for every open interval $I$ containing $x$ we have $I \cap (A \setminus \{x\}) \neq \emptyset$. Thus, for every open interval $I$ containing $x$ there exists a $p \in I \cap (A \setminus \{x\})$. Since $A \subseteq B$ we have also $A \setminus \{x\} \subseteq B \setminus \{x\}$. From this and the preceding sentence we conclude that for every open interval $I$ containing $x$ there exists a $p \in I \cap (B \setminus \{x\})$. So, for every open interval $I$ containing $x$ we have $I \cap (B \setminus \{x\}) \neq \emptyset$. Therefore, by Thm. 2.3, $x \sim B \setminus \{x\}$. Thus, by definition, $x \in B^\ell$. We proved that for every $x \in \mathbb{R}$, if $x \in A^\ell$ then $x \in B^\ell$. So, $A^\ell \subseteq B^\ell$.

**Method II.** First note that $A \subseteq B$ implies that $A \setminus \{x\} \subseteq B \setminus \{x\}$. So, by Exercise 2.10,

\[ (\star) \ (A \setminus \{x\})^\sim \subseteq (B \setminus \{x\})^\sim. \]

Now, let $x \in A^\ell$. Then, by definition, $x \sim A \setminus \{x\}$. Therefore, by $(\star)$, $x \in (A \setminus \{x\})^\sim \subseteq (B \setminus \{x\})^\sim$. So, $x \in (B \setminus \{x\})^\sim$. Thus, $x \sim B \setminus \{x\}$, that is, $x \in B^\ell$.

We proved that for every $x \in \mathbb{R}$, if $x \in A^\ell$ then $x \in B^\ell$. So, $A^\ell \subseteq B^\ell$. 

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Ex. 2.32. If $A \subset \mathbb{R}$ then $(A^\ell)^\ell \subset A^\ell$. Must $(A^\ell)^\ell = A^\ell$?

**Proof of $(A^\ell)^\ell \subset A^\ell$.** Let $x \in (A^\ell)^\ell$. We must show that $x \in A^\ell$, that is, that $x \sim A \setminus \{x\}$, which is equivalent to

\[(\ast) \ I \cap (A \setminus \{x\}) \neq \emptyset \text{ for every open interval } I \text{ containing } x.\]

To prove this last condition fix an open interval $I$ containing $x$. Since $x \in (A^\ell)^\ell$, we have $x \sim A^\ell \setminus \{x\}$ and so, by Thm. 2.3, $I \cap (A^\ell \setminus \{x\}) \neq \emptyset$. Thus, there exists a $z \in I \cap (A^\ell \setminus \{x\})$. Let $J$ be an open interval such that $z \in J \subset I \setminus \{x\}$. Since $z \in A^\ell$ we have $z \sim A \setminus \{z\}$ and, by Thm. 2.3, $J \cap (A \setminus \{z\}) \neq \emptyset$. In particular, there is a $t \in J \cap (A \setminus \{z\})$. Now, we have that $t \in J \subset I \setminus \{x\}$ and $t \in A$. So, $t \in I$ and $t \in A \setminus \{x\}$. Thus, $I \cap (A \setminus \{x\}) \neq \emptyset$. Condition $(\ast)$ has been proved, what finishes the proof.

**Must we have $(A^\ell)^\ell = A^\ell$?** NO.

Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then, by Exercise 2.28, we have $A^\ell = \{0\}$ and $(A^\ell)^\ell = \{0\}^\ell = \emptyset$. Thus, for this $A$ we have

\[(A^\ell)^\ell = \emptyset \neq \{0\} = A^\ell.\]
Ex. 3.7. Prove that \( \lim_{x \to p} \sqrt{x} = \sqrt{p} \) for all \( p \geq 0 \).

**Proof. Method I.** Case \( p > 0 \): For every \( x \geq 0 \) we have

\[
|\sqrt{x} - \sqrt{p}| = \frac{|x - p|}{\sqrt{x} + \sqrt{p}} \leq \frac{|x - p|}{\sqrt{p}}.
\]

Thus, if for \( \varepsilon > 0 \) we put \( \delta = \sqrt{p} \varepsilon \) then \( \delta > 0 \) and for every \( x \geq 0 \) for which \( |x - p| < \delta \) we have

\[
|\sqrt{x} - \sqrt{p}| \leq \frac{|x - p|}{\sqrt{p}} < \frac{\delta}{\sqrt{p}} = \sqrt{p} \varepsilon = \varepsilon.
\]

So, \( \lim_{x \to p} \sqrt{x} = \sqrt{p} \) for all \( p > 0 \).

Case \( p = 0 \): For \( \varepsilon > 0 \) put \( \delta = \varepsilon^2 \). Then \( \delta > 0 \) and for every \( x \geq 0 \) for which \( |x - 0| < \delta \) we have \( \sqrt{x} < \sqrt{\delta} = \varepsilon \), so that \( |\sqrt{x} - \sqrt{0}| = \sqrt{x} < \varepsilon \). So, we proved that \( \lim_{x \to 0} \sqrt{x} = \sqrt{0} \) for \( p = 0 \).

**Method II.** First notice that for every \( a \geq b \geq 0 \) we have \( a - b \leq a + b \).

Multiplying both sides by \( a - b \geq 0 \) we obtain that \( (a - b)^2 \leq a^2 - b^2 \). Taking square root from both sides we obtain that \( a - b \leq \sqrt{a^2 - b^2} \) for all \( a \geq b \geq 0 \).

Replacing \( a \) and \( b \) when \( b \geq a \geq 0 \) we obtain a rule:

\[
|a - b| \leq \sqrt{|a^2 - b^2|} \quad \text{for all} \ a, b \geq 0.
\]

Substituting in the above \( a = \sqrt{x} \) and \( b = \sqrt{p} \) we obtain the inequality

\[
|\sqrt{x} - \sqrt{p}| \leq \sqrt{|x - p|} \quad \text{for all} \ x, p \geq 0.
\]

Coming back to the limit. Fix \( p \geq 0 \) and \( \varepsilon > 0 \). Then for \( \delta = \varepsilon^2 \) we have \( \delta > 0 \) and for every \( x \geq 0 \), if \( |x - p| < \delta \) then

\[
|\sqrt{x} - \sqrt{p}| \leq \sqrt{|x - p|} < \sqrt{\delta} = \varepsilon.
\]

So, we proved that \( \lim_{x \to p} \sqrt{x} = \sqrt{p} \) for all \( p \geq 0 \).
Ex. 3.18. Find \( \lim_{x \to 4} \frac{|x-4|}{x-4} \) (if it exists).

**Proof.** Notice that for \( x < 4 \) we have \( \frac{|x-4|}{x-4} = \frac{-(x-4)}{x-4} = -1 \). Thus
\[
\lim_{x \to 4^-} \frac{|x-4|}{x-4} = \lim_{x \to 4^-} -1.
\]
But the function \( f(x) = 0 \cdot x + (-1) = -1 \) is continuous by Theorem 2.23. Hence, by Theorem 3.12,
\[
\lim_{x \to 4} -1 = \lim_{x \to 4} f(x) = f(4) = 0 \cdot 4 - 1 = -1.
\]
Since \( \lim_{x \to 4} -1 \) exists, we have also, by Theorem 3.16, that
\[
\lim_{x \to 4^-} \frac{|x-4|}{x-4} = \lim_{x \to 4^-} -1 = -1.
\]
Similarly since \( g(x) = 0 \cdot x + 1 = 1 \) is continuous by Theorem 2.23, we have
\[
\lim_{x \to 4^+} \frac{|x-4|}{x-4} = \lim_{x \to 4^+} 1 = \lim_{x \to 4^+} g(x) = g(4) = 0 \cdot 4 + 1 = 1.
\]
Now, \( \lim_{x \to 4^+} \frac{|x-4|}{x-4} = 1 \neq -1 = \lim_{x \to 4^-} \frac{|x-4|}{x-4} \) so, by Theorem 3.16, \( \lim_{x \to 4} \frac{|x-4|}{x-4} \) does not exist.
Ex. 3.20. Find \( \lim_{x \to 1} \frac{x - 1}{x^2 + x - 2} \) (if it exists).

**Proof.** We will show that \( \lim_{x \to 1^+} \frac{x - 1}{x^2 + x - 2} = \frac{1}{3} \) and \( \lim_{x \to 1^-} \frac{x - 1}{x^2 + x - 2} = -\frac{1}{3} \). Since these two limits are not equal, by Theorem 3.16 we will conclude that \( \lim_{x \to 1} \frac{x - 1}{x^2 + x - 2} \) does not exist.

First notice that for \( x < 1 \) we have \( \frac{x - 1}{x^2 + x - 2} = \frac{x - 1}{|x - 1||x + 2|} = -\frac{1}{3} \). Now, if \( |x - 1| < \delta \leq 0.5 \) then \( x + 2 > 0 \) and so \( \frac{x - 1}{x^2 + x - 2} = -\frac{1}{x + 2} \). In order to prove that \( \lim_{x \to 1^-} \frac{x - 1}{x^2 + x - 2} = -\frac{1}{3} \) it is enough to show that for every \( \varepsilon > 0 \) there exists a \( \delta \in (0,0.5) \) such that for all \( x < 1 \) for which \( |x - 1| < \delta \) the number

\[
\left| \frac{x - 1}{x^2 + x - 2} - \frac{1}{3} \right| = \left| \frac{x - 1}{x + 2} + \frac{1}{3} \right| = \frac{|x - 1|}{3|x + 2|}
\]

is less than \( \varepsilon \). But \( |x - 1| < \delta \leq 0.5 \) implies that \( x + 2 > 0.5 \) so \( \frac{1}{x + 2} \leq 2 \) and \( \frac{|x - 1|}{3|x + 2|} \leq \frac{2}{3} |x - 1| \). Thus if \( \delta = \min \{0.5, \frac{3}{2} \varepsilon \} \) then \( |x - 1| < \delta \leq 0.5 \) implies that

\[
\left| \frac{x - 1}{x^2 + x - 2} - \frac{1}{3} \right| = \left| \frac{x - 1}{3|x + 2|} \right| \leq \frac{2}{3} |x - 1| < \frac{2}{3} \cdot \frac{3}{2} \varepsilon = \varepsilon.
\]

This proves that \( \lim_{x \to 1^-} \frac{x - 1}{x^2 + x - 2} = -\frac{1}{3} \).

Similarly, to prove that \( \lim_{x \to 1^+} \frac{x - 1}{x^2 + x - 2} = \frac{1}{3} \) choose an \( \varepsilon > 0 \), define \( \delta = \min \{0.5, \frac{3}{2} \varepsilon \} \) and notice that for \( x > 1 \) such that \( |x - 1| < \delta \) we have \( \frac{x - 1}{x^2 + x - 2} = \frac{x - 1}{|x - 1||x + 2|} = \frac{1}{x + 2} = \frac{1}{x + 2} \) and so

\[
\left| \frac{x - 1}{x^2 + x - 2} - \frac{1}{3} \right| = \left| \frac{1}{x + 2} - \frac{1}{3} \right| = \left| \frac{x - 1}{3|x + 2|} \right| \leq \frac{2}{3} |x - 1| < \frac{2}{3} \cdot \frac{3}{2} \varepsilon = \varepsilon.
\]

This proves that \( \lim_{x \to 1^+} \frac{x - 1}{x^2 + x - 2} = \frac{1}{3} \). So, \( \lim_{x \to 1} \frac{x - 1}{x^2 + x - 2} \) does not exist.
Ex. 3.9. Assume that \( \lim_{x \to p} f(x) = \sqrt{82} - 9 \), where \( p \) is a limit point of the domain \( X \) of \( f \). Prove that there is a \( \delta > 0 \) such that \( f(x) > 0 \) for all \( x \in X \setminus \{p\} \) such that \( |x - p| < \delta \). If \( p \in X \), must \( f(p) > 0 \)?

Proof. First notice that \( L = \sqrt{82} - 9 > 0 \).

Since \( \lim_{x \to p} f(x) = L \) by the definition of the limit we have:

for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( |f(x) - L| < \varepsilon \)

provided \( x \in X \setminus \{p\} \) is such that \( |x - p| < \delta \).

Since the expression \( |f(x) - L| < \varepsilon \) is equivalent to \( L - \varepsilon < f(x) < L + \varepsilon \), in particular we know that

for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( L - \varepsilon < f(x) \)

provided \( x \in X \setminus \{p\} \) is such that \( |x - p| < \delta \).

Since this is true for EVERY \( \varepsilon > 0 \) it is also true for \( \varepsilon = L > 0 \), for which the above transforms to:

there exists a \( \delta > 0 \) such that \( 0 = L - L < f(x) \)

provided \( x \in X \setminus \{p\} \) is such that \( |x - p| < \delta \).

But this is precisely what we had to prove.

If \( p \in X \), must \( f(p) > 0 \)? NO. No, the value of the limit has nothing to do with the value of \( f \) at \( p \). For example, let \( f(x) = \sqrt{82} - 9 \) for all \( x \neq p \) and \( f(p) = -1 \). Then clearly \( f(p) < 0 \) while

\[
\lim_{x \to p} f(x) = \lim_{x \to p} (\sqrt{82} - 9) = \sqrt{82} - 9,
\]

since \( g(x) = \sqrt{82} - 9 \) is a linear function.
Ex. 4.14. Give an example of two functions $f, g: \mathbb{R} \to \mathbb{R}$ such that for some point $p \in \mathbb{R}$, $\lim_{x \to p} (f \cdot g)(x)$ exists but $\lim_{x \to p} f(x)$ and $\lim_{x \to p} g(x)$ do not exist.

SOLUTION. Let $f(x) = g(x) = \frac{|x-4|}{x-4}$ for all $x \in \mathbb{R} \setminus \{4\}$. By Exercise 3.18 we know that $\lim_{x \to 4} \frac{|x-4|}{x-4}$ does not exist, so neither $\lim_{x \to 4} f(x)$ nor $\lim_{x \to 4} g(x)$ exist. But

$$\lim_{x \to p} (f \cdot g)(x) = \lim_{x \to p} \frac{|x-4|}{x-4} \cdot \frac{|x-4|}{x-4} = \lim_{x \to p} \frac{(x-4)^2}{(x-4)^2} = \lim_{x \to p} 1 = 1$$

clearly exists.

Ex. 4.15. Are there two functions $f, g: \mathbb{R} \to \mathbb{R}$ such that for some point $p \in \mathbb{R}$, $\lim_{x \to p} (f \cdot g)(x)$ and $\lim_{x \to p} f(x)$ both exist but $\lim_{x \to p} g(x)$ does not exist?

SOLUTION. YES. Again let $g(x) = \frac{|x-4|}{x-4}$. So, as above, $\lim_{x \to 4} g(x)$ does not exist. But if $f(x) = 0$ for all $x \in \mathbb{R}$ then $f(x) = 0 \cdot x + 0$ is a linear function, so continuous, and the limits

$$\lim_{x \to p} (f \cdot g)(x) = \lim_{x \to p} f(x) = \lim_{x \to p} 0 = 0$$

clearly exist.
**Bonus Exercise.** Let \( g: X \to \mathbb{R} \). Show that if \( \lim_{x \to p} g(x) = M \neq 0 \) exists then there is a \( \delta_1 \) such that \( |g(x)| > \frac{|M|}{2} \) for all \( x \in X \setminus \{p\} \) such that \( |x - p| < \delta_1 \).

**Solution.** Let \( \varepsilon = \frac{|M|}{2} \). By the definition of the limit there exists a \( \delta_1 \) such that for all \( x \in X \setminus \{p\} \) with \( |x - p| < \delta_1 \) we have

(a) \( |g(x) - M| < \frac{|M|}{2} \).

But (a) implies that \( -\frac{|M|}{2} < -|g(x) - M| \), so adding to this \( |M| \), and using the triangle inequality \( |a + b| - |b| \leq |a| \), with \( a = g(x) \) and \( b = M - g(x) \) we get

\[ \frac{|M|}{2} = |M| - \frac{|M|}{2} < |M| - |g(x) - M| = |g(x) + (M - g(x))| - |M - g(x)| \leq |g(x)|. \]

So \( \delta_1 \) that insures (a) also guarantees that \( |g(x)| > \frac{|M|}{2} \).

**Comment to Bonus Exercise.** In fact, if \( \lim_{x \to p} g(x) = M \) then inequality \( ||a| - |b|| \leq |a - b| \) allows us easily to conclude that \( \lim_{x \to p} |g(x)| = |M| \), without the use of Squeeze Theorem.

**Proof.** \( \lim_{x \to p} g(x) = M \) is equivalent, by the definition of the limit, to

for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for every \( x \in X \setminus \{p\} \) with \( |x - p| < \delta \) we have \( |g(x) - M| < \varepsilon \).

But, using the above inequality with \( a = g(x) \) and \( b = M \), we conclude that \( ||g(x)| - |M|| \leq |g(x) - M| \). So, the above statement implies that

for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for every \( x \in X \setminus \{p\} \) with \( |x - p| < \delta \) we have \( ||g(x)| - |M|| < \varepsilon \).

But, by the definition of the limit, this means that \( \lim_{x \to p} |g(x)| = |M| \).
Ex. 4.23. Are there two functions \( f, g : \mathbb{R} \to \mathbb{R} \), \( g(x) \neq 0 \) for all \( x \in \mathbb{R} \), such that for some point \( p \in \mathbb{R} \), \( \lim_{x \to p} \frac{f(x)}{g(x)} \) and \( \lim_{x \to p} f(x) \) both exist but \( \lim_{x \to p} g(x) \) does not exist?

Solution. YES. The same example works that I gave for Exercise 4.15. If \( g(x) = \frac{|x-4|}{x-4} \) then \( \lim_{x \to 4} g(x) \) does not exist. But if \( f(x) = 0 \) for all \( x \in \mathbb{R} \) then also \( \frac{f}{g}(x) = 0 \) and we have

\[
\lim_{x \to p} \frac{f}{g}(x) = \lim_{x \to p} f(x) = 0
\]

so the two limits clearly exist.

Ex. 4.24. Are there two functions \( f, g : \mathbb{R} \to \mathbb{R} \), \( g(x) \neq 0 \) for all \( x \in \mathbb{R} \), such that for some point \( p \in \mathbb{R} \), \( \lim_{x \to p} \frac{f(x)}{g(x)} \) and \( \lim_{x \to p} g(x) \) both exist but \( \lim_{x \to p} f(x) \) does not exist?

Solution. The answer is NO. By Theorem 4.9 we have

\[
\lim_{x \to p} f(x) = \lim_{x \to p} \left( \frac{f}{g} \right)(x) = \left( \lim_{x \to p} g(x) \right) \cdot \left( \lim_{x \to p} f(x) \right)
\]

So, \( \lim_{x \to p} f(x) \) does exist.
Ex. 4.33. For any two functions $f, g : \mathbb{R} \to \mathbb{R}$ define the maximum function $f \vee g$, written as $f \vee g$, and the minimum function $f \wedge g$, written as $f \wedge g$, as follows: for each $x \in \mathbb{R}$,

$$(f \vee g)(x) = \max\{f(x), g(x)\}, \quad (f \wedge g)(x) = \min\{f(x), g(x)\}.$$ 

Prove that if $f \vee g$ and $f \wedge g$ are continuous at $p$ then $f$ and $g$ are continuous at $p$.

(Hint: What is $\frac{x+y}{2} + \frac{|x-y|}{2}$ for real numbers $x$ and $y$?)

Proof. First notice that

\((*)\) $\max\{x, y\} = \frac{x+y}{2} + \frac{|x-y|}{2}$ for all $x, y \in \mathbb{R}$.

Indeed, if $x \geq y$ then $\frac{x+y}{2} + \frac{|x-y|}{2} = \frac{x+y}{2} + \frac{x-y}{2} = x = \max\{x, y\}$. Similarly, if $x \leq y$ then $\frac{x+y}{2} + \frac{|x-y|}{2} = \frac{x+y}{2} + \frac{y-x}{2} = y = \max\{x, y\}$.

From $(*)$ we obtain that

$$(f \vee g)(x) = \frac{f(x) + g(x)}{2} + \frac{|f(x) - g(x)|}{2}.$$ 

Next notice that function $h_0(x) = |x|$ is continuous. This follows from Exercise 4.31, which was proved in Comment to Bonus Exercise.

Now, by Corollary 4.4, function $h_1(x) = f(x) - g(x)$ is continuous at $p$, and so is, by Theorem 4.28, the composition function $h_2(x) = h_0 \circ h_1(x) = |f(x) - g(x)|$. Therefore, by Corollary 4.6, the function

$$h_3(x) = f(x) + g(x) + h_2(x) = f(x) + g(x) + |f(x) - g(x)|$$

is continuous at $p$ as a sum of three functions with this property. Finally, since $h_4(x) = \frac{1}{2}$ is continuous, as a linear function, by Corollary 4.10 the function

$$h_5(x) = h_4(x) \cdot h_3(x) = \frac{1}{2}(f(x) + g(x) + |f(x) - g(x)|) = (f \vee g)(x)$$

is continuous at $p$, finishing the argument for $f \vee g$.

The argument for $f \wedge g$ is very similar, using the fact that

$$(f \wedge g)(x) = \frac{f(x) + g(x)}{2} - \frac{|f(x) - g(x)|}{2}.$$
Ex. 4.35. Find \( \lim_{x \to 0} x \sin \left( \frac{1}{x} \right) \).

SOLUTION. Note that \( |x \sin \left( \frac{1}{x} \right)| = |x| |\sin \left( \frac{1}{x} \right)| \leq |x| \) since \( |\sin \left( \frac{1}{x} \right)| \leq 1 \) and \( |x| \geq 0 \). Thus, \(-|x| \leq x \sin \left( \frac{1}{x} \right) \leq |x| \) for all \( x \). Let us put \( g(x) = -|x| \), \( f(x) = x \sin \left( \frac{1}{x} \right) \), and \( h(x) = |x| \). Then, \( g(x) \leq f(x) \leq h(x) \) for all \( x \). Also, since functions \( g \) and \( h \) are continuous, we have \( \lim_{x \to 0} g(x) = -|0| = 0 \) and \( \lim_{x \to 0} h(x) = |0| = 0 \). Therefore, by Squeeze Theorem,

\[
\lim_{x \to 0} x \sin \left( \frac{1}{x} \right) = \lim_{x \to 0} f(x) = 0.
\]
**Ex. 5.3.** The restriction of a continuous function is continuous; in fact, if \( f : X \to \mathbb{R} \) is continuous at \( p \in X \) and \( p \in X' \subset X \) then \( f \upharpoonright X' \) is continuous at \( p \).

**SOLUTION.** We know that \( f \) is continuous at \( p \in X \), that is (by the definition) that

\[
(*) \quad \text{for every } A \subset X \text{ if } p \sim A \text{ then } f(p) \sim f[A].
\]

We need to prove that \( f \upharpoonright X' \) is continuous at \( p \), that is, that

\[
\text{for every } A \subset X' \text{ if } p \sim A \text{ then } (f \upharpoonright X')(p) \sim (f \upharpoonright X')[A].
\]

So, take an \( A \subset X' \) such that \( p \sim A \). To show the above condition we need to prove that \((f \upharpoonright X')(p) \sim (f \upharpoonright X')[A].\)

Since we have also \( A \subset X \) (as \( A \subset X' \subset X \)) by (*) applied to this set we have \( f(p) \sim f[A]. \) But \((f \upharpoonright X')(p) = f(p)\) and

\[
(f \upharpoonright X')[A] = \{(f \upharpoonright X')(a) : a \in A\} = \{f(a) : a \in A\} = f[A].
\]

Therefore, \((f \upharpoonright X')(p) = f(p) \sim f[A] = (f \upharpoonright X')[A]\), and we have the desired relation \((f \upharpoonright X')(p) \sim (f \upharpoonright X')[A]\).

**Ex. 5.5.** Let \( f(x) = \frac{35}{\sqrt{2x^{14} + 5x^{10} + 9x^8 + 3x^4 + 7}}. \) Show that \( f(x) = \frac{10}{7} \) for some \( x \in [0,1] \).

**PROOF.** We will use the Intermediate Value Theorem to show this. Forst notice that \( f \) is a rational function, so it is continuous on its domain, which consists of a set of all points in at which the denominator is not equal to 0. However, we need to know that \( f \) is continuous on the interval \([0,1]\). This follows easily from the comment above when we notice that the denominator of \( f: \sqrt{2x^{14} + 5x^{10} + 9x^8 + 3x^4 + 7} \geq \sqrt{2} \cdot 0 + 5 \cdot 0 + 9 \cdot 0 + 3 \cdot 0 + 7 = 7 > 0 \). Thus, \( f \) is continuous on \([0,1]\).

Next note that \( f(1) = \frac{35}{\sqrt{2+5+9+3+7}} = \frac{35}{\sqrt{24}} < \frac{35}{24.5} = \frac{10}{7}. \) We also have \( f(0) = \frac{35}{7} = 5 > \frac{10}{7}. \) Thus,

\[
f(1) < \frac{10}{7} < f(0) \text{ and } f \text{ is continuous on } [0,1].
\]

Since all the assumptions of the Intermediate Value Theorem are satisfied we conclude that \( f(x) = \frac{10}{7} \) for some \( x \in [0,1] \).
Ex. 5.14. Finish the proof of the Maximum-Minimum Theorem proving that $f$ has a minimum value.

**Proof.** We already know that every continuous $f: [a, b] \to \mathbb{R}$ has a maximum value, that is, that

$$\text{there exists a } c \in [a, b] \text{ such that } f(c) \geq f(x) \text{ for all } x \in [a, b].$$

To show that $f$ has also a minimum notice that $g: [a, b] \to \mathbb{R}$ defined by $g(x) = -f(x)$ is continuous. So, by the above proved statement, there exists a $d \in [a, b]$ such that $g(d) \geq g(x)$ for all $x \in [a, b]$. Thus, for all $x \in [a, b]$ we have $-f(d) = g(d) \geq g(x) = -f(x)$, that is, $f(d) \leq f(x)$. But this means that $f(d)$ is a minimum value for $f$ on $[a, b]$.

Ex. 5.19. True or false: If $f: (a, b] \to \mathbb{R}$ is continuous, then $f$ has a maximum value or $f$ has a minimum value.

**Solution.** False. Define $f: (0, 1] \to \mathbb{R}$ by $f(x) = \frac{\sin(1/x)}{x}$. Then for every $n \in \mathbb{N}$ we have

- $x_n = \frac{1}{2n\pi + \pi/2} \in (0, 1]$ and $f(x_n) = 2n\pi + \pi/2$;
- $y_n = \frac{1}{2n\pi - \pi/2} \in (0, 1]$ and $f(x_n) = -(2n\pi - \pi/2)$.

Thus, $f$ has arbitrary large values and arbitrary small values. So $f$ has neither a maximum value nor a minimum value.
Ex. 6.6. Find the points at which the function \( f(x) = \sqrt{x} \) is differentiable and find its derivative at these points.

Proof. First assume that \( x > 0 \). Then

\[
f'(x) = \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}} = \lim_{h \to 0} \frac{(x + h) - x}{h(\sqrt{x + h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}} = \frac{1}{\sqrt{x + 0} + \sqrt{x}},
\]

where the last equation follows from the continuity of \( g(h) = \frac{1}{\sqrt{x + h} + \sqrt{x}} \).

(To see that \( g \) is continuous, note that \( g_0(h) = \sqrt{x + h} \) is continuous, as a composition of a linear function \( g_1(h) = x + h \) and \( g_2(x) = \sqrt{x} \). Thus, \( g_3(h) = \sqrt{x + h} + \sqrt{x} \) is continuous as a sum of a continuous \( g_0 \) and a constant function \( g_4(h) = \sqrt{x} \). So, \( g \) is continuous as a reciprocal of a continuous function \( g_4 \).) From this we obtain that \( f \) is differentiable for \( x > 0 \) and for such \( x \) we have \( f'(x) = \frac{1}{2\sqrt{x}} \).

Next, we will show that \( f \) is not differentiable at \( x = 0 \). So, consider the limit

\[
\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{\sqrt{h}}{h} = \lim_{h \to 0} \frac{1}{\sqrt{h}}
\]

and notice that \( \lim_{h \to 0} \frac{1}{\sqrt{h}} \neq L \) for every \( L \in \mathbb{R} \). So, fix an \( L \in \mathbb{R} \). We need to show that the following is false:

\[
\forall \varepsilon > 0 \exists \delta > 0 \forall h \geq 0 \left( 0 < |h| < \delta \Rightarrow \left| \frac{1}{\sqrt{h}} - L \right| < \varepsilon \right),
\]

that is, that

\[
\exists \varepsilon > 0 \forall \delta > 0 \exists h \in (0, \delta) \left| \frac{1}{\sqrt{h}} - L \right| \geq \varepsilon.
\]

So, fix an \( \varepsilon = 1 \). For every \( \delta > 0 \) we need to find an \( h \in (0, \delta) \) for which

\[
\left| \frac{1}{\sqrt{h}} - L \right| \geq 1.
\]

So, it is enough to find an \( h \) with \( \frac{1}{\sqrt{h}} - L \geq 1 \). Since \( \frac{1}{\sqrt{h}} \geq |L| + 1 \) implies this inequality, it is enough to take any \( h \leq \frac{1}{(|L| + 1)^2} \).

Thus, \( h = \min \left\{ \frac{1}{(|L| + 1)^2}, \frac{\delta}{2} \right\} \) does the job.
Ex. 6.11. Assume that $f$ is defined on an open interval $I$ and that $f$ is differentiable at some point $p \in I$. Find

$$L = \lim_{h \to 0} \frac{f(p + h) - f(p - h)}{h}.$$ 

**Solution.** We have

$$L = \lim_{h \to 0} \frac{f(p + h) - f(p - h)}{h} = \lim_{h \to 0} \frac{f(p + h) - f(p) + f(p) - f(p - h)}{h}$$

$$= \lim_{h \to 0} \frac{f(p + h) - f(p)}{h} + \lim_{h \to 0} \frac{f(p) - f(p - h)}{h}$$

$$= f'(p) + \lim_{h \to 0} \frac{f(p) - f(p - h)}{h}.$$ 

Substituting in the last limit $h = -t$ (see Exercise 6.13) we obtain

$$L = f'(p) + \lim_{t \to 0} \frac{f(p) - f(p + t)}{-t} = f'(p) + \lim_{t \to 0} \frac{f(p + t) - f(p)}{t} = f'(p) + f'(p).$$

So, $L = 2f'(p)$. 

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Ex. 6.19. Are there constants $a$ and $b$ such that the function $f$ given by

$$f(x) = \begin{cases} x^2 + 5 & \text{if } x \leq 1 \\ ax + b & \text{if } x > 1 \end{cases}$$

is differentiable?

**Solution.** Let $f_0(x) = x^2 + 5$ and $f_1(x) = ax + b$. First note that, by Example 6.2, $f_1$ is differentiable with $f_1'(x) = a$, while $f_0$ is differentiable, since

$$f_0'(x) = \lim_{h \to 0} \frac{[(x + h)^2 + 5] - [x^2 + 5]}{h} = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = 2x.$$ 

This implies that $f$ is differentiable for at all points $x$, except possible at $x = 1$. In order to make $f$ differentiable at $x = 1$ we need to choose appropriate $a$ and $b$.

For $f$ to be differentiable at $x = 1$ it must be continuous there. Since $f_0$ is continuous, we have $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} f_0(x) = f_0(1) = f(1)$. Thus, we need also to ensure that $\lim_{x \to 1^+} f(x) = f(1)$. This leads to the condition

$$a + b = \lim_{x \to 1^+} f_1(x) = \lim_{x \to 1^+} f(x) = f(1) = 1^2 + 5 = 6.$$ 

Assuming that this holds, we get $f \upharpoonright [1, \infty) = f_1 \upharpoonright [1, \infty)$. In particular, we get

$$f'_+(1) = (f \upharpoonright [1, \infty))'_+(1) = (f_1 \upharpoonright [1, \infty))'_+(1) = (f_1)'(1) = a.$$ 

We also have

$$f'_-(1) = (f \upharpoonright (-\infty, 1])'_+(1) = (f_0 \upharpoonright (-\infty, 1])'_+(1) = (f_0)'(1) = 2.$$ 

Thus, we need to have $a = f'_+(1) = f'_-(1) = 2$. Therefore, $f$ is differentiable if and only if $a = 2$ and $a + b = 6$ or, equivalently, when $a = 2$ and $b = 4$. 

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Ex. 7.9. Assume that \( \frac{f}{g}(x) = x^2 + 2x \), where \( f \) and \( g \) are differentiable functions such that \( f(2) = 2 \) and \( f'(2) = 3 \). Find \( g'(2) \).

SOLUTION. The quotient rule gives us that

\[
\left( \frac{f}{g} \right)'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2}. \tag{1}
\]

Since, by Theorem 6.14 and Examples 6.3 and 6.2 we have

\[
\left( \frac{f}{g} \right)'(x) = \frac{d}{dx} (x^2) + \frac{d}{dx} (2x) = 2x + 2
\]

we get that \( \left( \frac{f}{g} \right)'(2) = 2 \cdot 2 + 2 = 6 \). Also \( g(x) = \frac{f(x)}{x^2 + 2x} \), so \( g(2) = \frac{f(2)}{2^2 + 2 \cdot 2} = \frac{1}{4} \).

Substituting these to (1) we obtain

\[
6 = \frac{\frac{1}{4} \cdot 3 - 2g'(2)}{\left(\frac{1}{4}\right)^2}.
\]

Solving this for \( g'(2) \) we obtain \( g'(2) = \frac{3}{16} \).
Ex. 7.15. Prove that \((x^n)' = nx^{n-1}\) for each \(n = -1, -2, \ldots\).

**Proof.** Let \(n = -k\), where \(k = 1, 2, \ldots\). Then

\[
(x^n)' = \left(\frac{1}{x^k}\right)' = \frac{-kx^{k-1}}{[x^k]^2} = -kx^{-k-1} = nx^{n-1}
\]

finishing the proof.

Ex. 7.21. (BONUS) For which polynomials \(f\) and \(g\) the bogus formula 
\((f \cdot g)'(x) = f'(x) \cdot g'(x)\) holds?

**Answer:** This happens if and only if the product either one of the functions \(f\) or \(g\) is a constant equal zero, or when both functions are constant. (Or, equivalently, when \(f(x)g(x)\) is a constant function.)

To argue for this first notice that functions are as we described than the formula \((f \cdot g)'(x) = f'(x) \cdot g'(x)\) holds with both sides equal to 0. Thus, we need to argue that no other polynomials satisfy the formula. We will consider two cases.

**Case 1.** One of the function, say \(f(x)\), is a constant \(c \neq 0\). But then, by Theorem 7.12, \((f \cdot g)'(x) = (c \cdot g)'(x)\) is a non-zero polynomial, while \(f'(x) \cdot g'(x) = 0 \cdot g'(x) = 0\). So, \((f \cdot g)'(x) \neq f'(x) \cdot g'(x)\).

**Case 2.** Both functions \(f\) and \(g\) are not constant. Now, if for a polynomial \(w\) its degree is denoted by \(\deg(w)\) then \(\deg(f) = n > 0\) and \(\deg(g) = k > 0\). But then we have \(\deg(f \cdot g) = \deg(f) + \deg(g) = n + k\) and, by Theorem 7.12, \(\deg((f \cdot g)') = n + k - 1\), while \(\deg(f' \cdot g') = \deg(f') + \deg(g') = (n-1) + (k-1)\). Therefore \(\deg((f \cdot g)') \neq \deg(f' \cdot g')\) and so \((f \cdot g)'(x) \neq f'(x) \cdot g'(x)\).
Simpler proof of Lemma 8.2. We will prove only (1), thus we assume that $f(a) < f(b)$. We will start with showing that $f$ is strictly increasing on $[a, b]$, that is, that

- if $a \leq x_1 < x_2 \leq b$ then $f(x_1) < f(x_2)$.

To see it, first notice that $f(x_1) < f(b)$. Indeed, since $f$ is one-to-one, the only other possibility is that $f(b) < f(x_1)$. But then $f(a) < f(b) < f(x_1)$ and, by the Intermediate Value Theorem, there exists an $x \in (a, x_1)$ for which $f(x) = f(b)$, contradicting the assumption that $f$ is one-to-one, since $x \neq b$, as $x < x_1 < b$.

So $f(x_1) < f(b)$. To see that $f(x_1) < f(x_2)$ note that the only other possibility (since $f$ is one-to-one) is that $f(x_2) < f(x_1)$ and, by the Intermediate Value Theorem, there would exist an $x \in (x_2, b)$ for which $f(x) = f(x_1)$, contradicting the assumption that $f$ is one-to-one, since $x \neq x_1$, as $x_1 < x_2 < x$.

To finish the prove notice that since $f$ is strictly increasing on $[a, b]$ we must have $f[[a, b]] = [f(a), f(b)]$ and, by Exercise 8.1, $f^{-1}$ is strictly increasing on $[f(a), f(b)]$. This finishes the proof of (1).

Simpler proof of Theorem 8.4. We will prove the following fact which holds for every continuous one-to-one function $f: I \to \mathbb{R}$. This fact immediately implies the theorem.

- Let $c < d$ be arbitrary points from $I$. If $f(c) < f(d)$ then $f$ is strictly increasing on $I$. If $f(c) > f(d)$ then $f$ is strictly decreasing on $I$.

We will prove only the first implication, leaving the second as an exercise.

So, assume that $f(c) < f(d)$ and let $x_1 < x_2$ be arbitrary points from $I$. We need to show that $f(x_1) < f(x_2)$. To see this, let $a = \min\{c, d, x_1, x_2\}$ and $b = \max\{c, d, x_1, x_2\}$ and notice that $[a, b] \subset I$.

If $f(a) > f(b)$ then, by Lemma 8.2, $f$ is strictly decreasing on $[a, b]$. Since $a \leq c < d \leq b$, this implies that $f(c) > f(d)$, contradicting our assumption. Therefore we must have $f(a) < f(b)$. But then Lemma 8.2 implies $f$ is strictly increasing on $[a, b]$. Since $a \leq x_1 < x_2 \leq b$, this implies the desired inequality $f(x_1) < f(x_2)$.
Ex. 8.1. Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$ be a function. Prove that if $f$ is strictly increasing on $X$ then $f^{-1}$ is strictly increasing on $f[X]$; if $f$ is strictly decreasing on $X$ then $f^{-1}$ is strictly decreasing on $f[X]$;

PROOF. First assume that $f$ is strictly increasing on $X$. Notice that if $x_1, x_2 \in X$ are such that $f(x_1) < f(x_2)$ then $x_1 < x_2$.

Indeed, the inequality $x_2 \geq x_1$ is impossible since it implies $f(x_2) \leq f(x_1)$, as $f$ is strictly increasing on $X$, contradicting $f(x_1) < f(x_2)$.

Now, take $y_1 < y_2$ from $f[X]$. We need to show that $f^{-1}(y_1) < f^{-1}(y_2)$.

But $y_1, y_2 \in f[X]$ implies that there exist $x_1, x_2 \in X$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Thus, $y_1 < y_2$ translates to $f(x_1) < f(x_2)$. So, by the above remark, $x_1 < x_2$. Thus, $f^{-1}(y_1) = x_1 < x_2 = f^{-1}(y_2)$, finishing the proof of the first case.

Next assume that $f$ is strictly decreasing on $X$. We proceed similarly. First we note that if $x_1, x_2 \in X$ are such that $f(x_1) < f(x_2)$ then $x_1 > x_2$.

Indeed, the inequality $x_2 \geq x_1$ is impossible since it implies $f(x_2) \leq f(x_1)$, as $f$ is strictly decreasing on $X$, contradicting $f(x_1) < f(x_2)$.

Now, take $y_1 < y_2$ from $f[X]$. We need to show that $f^{-1}(y_1) > f^{-1}(y_2)$.

But $y_1, y_2 \in f[X]$ implies that there exist $x_1, x_2 \in X$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Thus, $y_1 < y_2$ translates to $f(x_1) < f(x_2)$. So, by the above remark, $x_1 > x_2$. Thus, $f^{-1}(y_1) = x_1 > x_2 = f^{-1}(y_2)$, finishing the proof.
Ex. 8.14. Let $f(x) = x^n$, where $n \in \mathbb{N}$. Show that if $n$ is even then $f$ is strictly increasing on $[0, \infty)$ and that if $n$ is odd then $f$ is strictly increasing on $\mathbb{R}$.

Proof. First note that

(*) $f(x) = x$ is strictly increasing on $\mathbb{R}$

since $f(a) = a < b = f(b)$ for any $a < b$.

As a first step we will show, by induction on $n$, that

(a) $f$ is strictly increasing on $[0, \infty)$ for any natural number $n$.

For $n = 1$ this follows from (*). So, assume that this holds for some $n$, that is, that $f(x) = x^n$ is strictly increasing on $[0, \infty)$. We need to show that $f(x) = x^{n+1}$ is strictly increasing on $[0, \infty)$. So, take $0 \leq a < b$ and note that $a^n < b^n$ by the inductive assumptions. Hence

$$f(a) = a^{n+1} = a^n a \leq b^n a < b^n b = b^{n+1} = f(b)$$

finishing the proof of (a).

Notice that this takes care of the problem for even $n$.

So, assume that $n$ is odd, say $n = 2k + 1$. We need to show that

(b) $f$ is strictly increasing on $\mathbb{R}$ for any $k = 0, 1, 2, \ldots$.

For $k = 0$ this was proved in (*). So, assume that $k > 0$ and take $a < b$.

First assume that $b \leq 0$. Then $0 \leq -b < -a$, so by part (a) we have $0^{2k} \leq (-b)^{2k} < (-a)^{2k}$ that is, $0 \leq b^{2k} < a^{2k}$. Thus

$$-f(b) = -b^{2k+1} = b^{2k}(-b) \leq a^{2k}(-b) < a^{2k}(-a) = -a^{2k+1} = -f(a)$$

and so $f(a) < f(b)$.

Combining this case and (a) we obtain that $f(a) < f(b)$ if either $a < b \leq 0$ or $0 \leq a < b$. But in the only other remaining case we have $a < 0 < b$ in which case $f(a) < f(0) < f(b)$, finishing the proof.
Ex. 8.26. The inverse cosine function has domain $[-1, 1]$ and range $[0, \pi]$. Prove that $(\cos^{-1})'(x) = \frac{-1}{\sqrt{1-x^2}}$.

**Proof.** Let $f(x) = \cos x$. Then, by Theorem 8.7,

$$(\cos^{-1})'(x) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{-\sin(\cos^{-1}(x))} = \frac{-1}{\sin(\cos^{-1}(x))}.$$  

Since $\cos^{-1}(x) \in [0, \pi]$ we have that $\sin(\cos^{-1}(x)) \geq 0$. Therefore, using the formula $|\sin x| = \sqrt{1-\cos^2 x}$ we get

$$\sin(\cos^{-1}(x)) = |\sin(\cos^{-1}(x))| = \sqrt{1-(\cos(\cos^{-1}(x)))^2} = \sqrt{1-x^2}.$$  

Combining two displays above gets us the required formula.
Ex. 9.20. Give examples of polynomials of degree 3 that have no critical point, only one critical point, and two critical points.

Solution. No critical point: \( f(x) = x^3 + 3x \). Then \( f'(x) = 3x^2 + 3 \) is always greater than 0, so it is never equal to 0.

Exactly one critical point: \( f(x) = x^3 \). Then \( f'(x) = 3x^2 \) is equal to 0 if and only if \( x = 0 \). Thus 0 is the only critical point of \( f \).

Two critical points: \( f(x) = x^3 - 3x \). Then \( f'(x) = 3x^2 - 3 = 3(x^2 - 1) \) is equal to 0 if and only if \( x = \pm 1 \). Thus -1 and 1 are the critical points of \( f \).

Ex. 9.21. Show that a polynomial of degree \( n > 0 \) has at most \( n \) roots.

Proof. We will prove this by induction on \( n > 0 \) that

\((I_n)\) each polynomial of degree \( n \) has at most \( n \) roots.

To see that \((I_1)\) is true take a polynomial \( f \) of degree 1, say \( f(x) = ax + b \), \( a \neq 0 \). Then \( f \) has precisely one root, \( -\frac{b}{a} \), so the statement is true.

Next, assume that for some \( n > 0 \) the statement \((I_n)\) is true. We need to show that \((I_{n+1})\) is true.

So, take a polynomial \( f \) of degree \( n+1 \) and by way of contradiction assume that it has \( n + 2 \) different roots, say \( r_0 < \cdots < r_{n+1} \). For \( i = 1, \cdots, n + 1 \) let \( I_i = [r_{i-1}, r_i] \) and note that

\( (*) \) there exists an \( x_i \in (r_{i-1}, r_i) \) such that \( f'(x_i) = 0 \).

Indeed, \( f \) is continuous on \( I_i \), so it must have a global maximum and global minimum at some points \( M \in I_i \) and \( m \in I_i \), respectively. If \( f(m) = f(M) \) then \( f \) is constant on \( I_i \) (can this really happen?) and any point \( x_i \in (r_{i-1}, r_i) \) satisfies \((*)\). So, assume that \( f(m) < f(M) \). Then one of these numbers must be different from 0 = \( f(r_{i-1}) = f(r_i) \). So, either \( m \) or \( M \) belongs to \( (r_{i-1}, r_i) \). Let \( x_i \) be such a number, that is, \( x_i \in \{m, M\} \cap (r_{i-1}, r_i) \). Then \( x_i \) is a critical point of \( f \) restricted to \( I_i \). Therefore, \( f'(x_i) = 0 \), since \( f \) is everywhere differentiable. Condition \((*)\) has been proved.

Now, \((*)\) implies that \( f' \) has at most \( n + 1 \) different roots \( x_1 < \cdots < x_{n+1} \).

Since, \( f' \) is a polynomial of degree \( n \) (see Theorem 7.12) this contradicts \((I_n)\).

Thus, our assumption that \( f \) has \( n + 2 \) different roots lead to contradiction, implying that \( f \) must have at most \( n + 1 \) roots. This concludes the proof that \((I_n)\) implies \((I_{n+1})\).

Now, the Induction Principle implies that \((I_n)\) holds for all \( n > 0 \).
Ex. 10.3. Define $f : [-2, 2] \to \mathbb{R}$ by $f(x) = x^3 - 3x + 3$. Find all numbers $p$ in $[-2, 2]$ that satisfy the conclusion of the Mean Value Theorem.

Solution. We need to find all numbers $p$ in $[-2, 2]$ for which $f'(p) = \frac{f(2) - f(-2)}{2 - (-2)}$. Since $f'(x) = 3x^2 - 3$, $f(2) = 8 - 6 + 3 = 5$, and $f(-2) = -8 + 6 + 3 = 1$ we need to solve the equation $3p^2 - 3 = \frac{5 - 1}{4}$, that is, $p^2 = \frac{4}{3}$. Since numbers $p = \pm \frac{2}{\sqrt{3}}$ are in $[-2, 2]$, they both consist of our solution.
Ex. 10.16. Let \( f \) be the function given by
\[
f(x) = \begin{cases} 
  x + 2 & \text{if } x < 0 \\
  x & \text{if } x \geq 0.
\end{cases}
\]
Is there a function \( g: \mathbb{R} \to \mathbb{R} \) such that \( g' = f \)?

**SOLUTION.** Assume that there exists such a function \( g \) and try to examine how it would need to look.

Clearly on \((-\infty, 0)\) function \( g \) must be on the form \( h_0(x) = \frac{1}{2}x^2 + 2x + C \).
(We use here the fact that \( \left( \frac{1}{2}x^2 + 2x \right)' = x + 2 \) and Theorem 10.9.) Since \( g \) must be continuous (as we like it to be differentiable) we must also have \( g(0) = \lim_{x \to 0^-} g(x) \). Therefore \( g(x) = h_0(0) \) on \((-\infty, 0]\). In particular, we would need to have
\[
g'_-(0) = \lim_{x \to 0^-} \frac{h_0(x) - h_0(0)}{x - 0} = h'_0(0) = 0 + 2 = 2.
\]

Similarly, on \((0, \infty)\) function \( g \) must be on the form \( h_1(x) = \frac{1}{2}x^2 + D \).
(We use here the fact that \( \left( \frac{1}{2}x^2 \right)' = x \) and Theorem 10.9.) Since function \( g \) must be continuous (as we like it to be differentiable) we must also have \( g(0) = \lim_{x \to 0^+} g(x) \). Therefore \( g(x) = h_1(0) \) on \([0, \infty)\). In particular, we would need to have
\[
g'_+(0) = \lim_{x \to 0^+} \frac{h_1(x) - h_1(0)}{x - 0} = h'_1(0) = 0.
\]

Thus, for our potential \( g \) we obtained that \( g'_-(0) \neq g'_+(0) \), so \( g \) cannot be differentiable. This shows that there is **NO** function \( g: \mathbb{R} \to \mathbb{R} \) such that \( g' = f \).
Ex. 12.26. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a nonnegative function that is integrable over \([a, b]\). Then \( \int_a^b f = 0 \) if and only if \( \inf f[I] = 0 \) for each open interval \( I \) in \([a, b]\).

**Solution.** “\( \Leftarrow \)” Assume that \( \inf f[I] = 0 \) for each open interval \( I \) in \([a, b]\). Then for every partition \( P = \{x_0, x_1, \ldots, x_n\} \) of \([a, b]\) we have that \( \inf f[x_{i-1}, x_i] = 0 \) and so

\[
L_f(P) = \sum_{i=1}^{n} m_i(f) \Delta x_i = \sum_{i=1}^{n} \Delta x_i \inf f[x_{i-1}, x_i] = 0.
\]

Therefore, \( \int_a^b f = \int_a^b f = \sup_{P \in P} L_f(P) = 0. \)

“\( \Rightarrow \)” By way of contradiction assume that there is an open interval \( I = (c, d) \) in \([a, b]\) with \( \inf f[I] = s > 0. \) Consider the partition \( P = \{a, c, d, b\} \). Since \( \inf f[x_{i-1}, x_i] \geq 0 \) for every \( i \), as \( f \) is a nonnegative, we have

\[
\int_a^b f \geq L_f(P) = \sum_{i=1}^{3} \Delta x_i \inf f[x_{i-1}, x_i] \geq \inf f[I] (d - c) = s (d - c) > 0.
\]

Thus, \( \int_a^b f = \int_a^b f \neq 0. \)

Ex. 12.30. Let \( f \) be differentiable on an interval \( I \). Show that if \( f' \) is bounded on \( I \) then \( f \) is uniformly continuous on \( I \).

**Proof.** Let \( L > 0 \) be such that \( |f'(x)| \leq L \) for every \( x \in I \). Notice that for every \( x_1 < x_2 \) from \( I \) we have

\[
|f(x_2) - f(x_1)| \leq L|x_2 - x_1|.
\]

Indeed, by the Mean Value Theorem there exists an \( x \in (x_1, x_2) \) such that \( \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x) \). Thus, \( \frac{|f(x_2) - f(x_1)|}{|x_2 - x_1|} = |f'(x)| \leq L \), giving the desired inequality.

To see that \( f \) is uniformly continuous on \( I \) fix an \( \varepsilon > 0 \) and put \( \delta = \varepsilon / L. \) To see that this \( \delta \) works, take \( x_1 < x_2 \) from \( I \) with \( |x_2 - x_1| < \delta \) and notice that then

\[
|f(x_2) - f(x_1)| \leq L|x_2 - x_1| \leq L\delta = \varepsilon.
\]
**Ex. 12.34.** Let $f : [a, b] \to \mathbb{R}$ be a function that is continuous at all but finitely many points. Show that $f$ is integrable on $[a, b]$.

**Proof.** First we will prove that

(*) if $f : [c, d] \to \mathbb{R}$ is bounded and continuous on $(c, d)$ then it is integrable.

To see this, take an $\epsilon > 0$. By Thm. 12.15 it is enough to find a partition $P$ of $[c, d]$ such that $U_P(f) - L_P(f) < \epsilon$. Since $f$ is bounded, there is an $M > 0$ such that $f[c, d] \subset [-M, M]$. Choose $p < q$ in $(c, d)$ close enough to $c$ and $d$, respectively, that $2M[(d - q) + (p - c)] < \epsilon / 2$. Since $f$ is continuous on $[p, q]$, it is $f \upharpoonright [p, q]$ is integrable. So, Thm. 12.15, there exists a partition $Q = \{x_0, x_1, \ldots, x_n\}$ of $[p, q]$ such that $U_Q(f) - L_Q(f) < \epsilon / 2$. Let $P = Q \cup \{c, d\}$. Then

$$L_f(P) = \inf f[c, p](p - c) + L_f(Q) + \inf f[q, d](d - q)$$

$$\geq -M(p - c) + L_f(Q) - M(d - q)$$

$$= L_f(Q) - M[(d - q) + (p - c)]$$

and similarly, $U_f(P) \leq U_f(Q) + M[(d - q) + (p - c)]$. Combining these two inequalities we get

$$U_P(f) - L_P(f) = U_Q(f) - L_Q(f) + 2M[(d - q) + (p - c)] < \epsilon / 2 + \epsilon / 2 = \epsilon.$$ 

This completes the proof for (*).

Now, to prove the main result, fix an $\epsilon > 0$. Thm. 12.15 it is enough to find a partition $R$ of $[c, d]$ such that $U_R(f) - L_R(f) < \epsilon$. So, let us choose $a = x_0 < x_1 < \cdots < x_n = b$ such that each point of discontinuity of $f$ is between these points. For every $i = 1, \ldots, n$ the function $f$ on $[x_{i-1}, x_i]$ satisfies the assumptions of (*). In particular, there exists a partition $R_i$ of $[x_{i-1}, x_i]$ such that $U_{R_i}(f) - L_{R_i}(f) < \epsilon / n$. Then $R = \bigcup_{i=1}^{n} R_i$ is a partition of $[a, b]$ and

$$U_R(f) - L_R(f) = \left( \sum_{i=1}^{n} U_{R_i}(f) \right) - \left( \sum_{i=1}^{n} L_{R_i}(f) \right) = \sum_{i=1}^{n} (U_{R_i}(f) - L_{R_i}(f))$$

is less than $n \cdot \epsilon / n = \epsilon$, finishing the proof.