Consequences of the Covering Property Axiom CPA


Under CPA we have $\mathfrak{c} = \omega_2$; $2^{\omega_1}$ can be arbitrarily large.

Real Functions

(F1) There exists a family $\mathcal{G}$ of uniformly continuous functions from $\mathbb{R}$ to $[0, 1]$ such that $|\mathcal{G}| = \omega_1$ and for every $S \in [\mathbb{R}]^\mathfrak{c}$ there exists a $g \in \mathcal{G}$ with $g[S] = [0, 1]$.

(F2) There exists a family $\mathcal{F}$ of less than continuum many $C^1$ functions from $\mathbb{R}$ to $\mathbb{R}$ such that the plain $\mathbb{R}^2$ is covered by functions from $\mathcal{F}$ and their inverses (i.e., each $f \in \mathcal{F}$ is used as a function on a horizontal axis and on a vertical axis.)

(F3) For every Borel function $f: \mathbb{R} \to \mathbb{R}$ there exists a family $\mathcal{F}$ of less than continuum many “$C^1$” functions (i.e., differentiable functions with continuous derivatives, where derivative can be infinite) whose graphs cover the graph of $f$.

(F4) For an arbitrary function $h$ from a subset $S$ of a Polish space $X$ onto a Polish space $Y$ there exists a uniformly continuous function $f$ from a subset of $X$ into $Y$ such that $|f \cap h| = \mathfrak{c}$. In particular,

- there is no Darboux Sierpiński-Zygmund function $f: \mathbb{R} \to \mathbb{R}$, that is, for every Darboux function $f: \mathbb{R} \to \mathbb{R}$ there is a subset $Y$ of $\mathbb{R}$ of cardinality $\mathfrak{c}$ such that $f \upharpoonright Y$ is continuous;

- for any function $h$ from a subset $S$ of $\mathbb{R}$ onto a perfect subset of $\mathbb{R}$ there exists a function $f \in \mathcal{C}^\infty_{\text{per}}$ such that $|f \cap h| = \mathfrak{c}$ and $f$ can be extended to a function $\bar{f} \in \mathcal{C}^1(\mathbb{R})$ such that either $\bar{f} \in C^1$ or $\bar{f}$ is an autohomeomorphism of $\mathbb{R}$ with $\bar{f}^{-1} \in C^1$.

(F5) For every Darboux function $g: \mathbb{R} \to \mathbb{R}$ there exists a continuous nowhere constant function $f: \mathbb{R} \to \mathbb{R}$ such that $f + g$ is Darboux.

(F6) There is a family $\mathcal{H}$ of $\omega_1$ pairwise disjoint perfect subsets of $\mathbb{R}$ such that $\mathcal{H} = \bigcup \mathcal{H}$ is a Hamel basis, that is, a linear basis of $\mathbb{R}$ over $\mathbb{Q}$. In particular,

- there is a non-measurable subset $X$ of $\mathbb{R}$ without the Baire property which is $\mathcal{N} \cap \mathcal{M}$-rigid, that is, such that $X \Delta (r + X) \in \mathcal{N} \cap \mathcal{M}$ for every $r \in \mathbb{R}$,

- there is a function $f: \mathbb{R} \to \mathbb{R}$ such that for every $h \in \mathbb{R}$ the difference function $\Delta_h(x) = f(x + h) - f(x)$ is Borel, but for every $\alpha < \omega_1$ there is an $h \in \mathbb{R}$ such that $\Delta_h$ is not of Borel class $\alpha$. 
(F7) There exists a discontinuous, almost continuous, and additive function \( f : \mathbb{R} \to \mathbb{R} \) whose graph is of measure zero.

(F8) There exists a Hamel basis \( H \) such that \( E^+(H) \) has measure zero, where \( E^+(A) \) is a linear combination of \( A \subset \mathbb{R} \) with non-negative rational coefficients.

(F9) For a Polish space \( X \) and uniformly bounded sequence \( \langle f_n : X \to \mathbb{R} \rangle_{n<\omega} \) of Borel measurable functions there are the sequences: \( \langle P_{\xi} : \xi < \omega_1 \rangle \) of compact subsets of \( X \) and \( \langle W_{\xi} \in [\omega]^\omega : \xi < \omega_1 \rangle \) such that \( X = \bigcup_{\xi<\omega_1} P_{\xi} \) and for every \( \xi < \omega_1 \):

\[
\langle f_n \restriction P_{\xi} \rangle_{n \in W_{\xi}} \text{ is a monotone uniformly convergent sequence of uniformly continuous functions.}
\]

(F10) Let \( X \) be an arbitrary set and \( f_n : X \to \mathbb{R} \) be a sequence of functions such that the set \( \{ f_n(x) : n < \omega \} \) is bounded for every \( x \in X \). Then there are the sequences: \( \langle P_{\xi} : \xi < \omega_1 \rangle \) of subsets of \( X \) and \( \langle W_{\xi} \in [\omega]^\omega : \xi < \omega_1 \rangle \) such that \( X = \bigcup_{\xi<\omega_1} P_{\xi} \) and for every \( \xi < \omega_1 \):

\[
\langle f_n \restriction P_{\xi} \rangle_{n \in W_{\xi}} \text{ is monotone and uniformly convergent.}
\]

Combinatorial Cardinal Characteristics

(C1) \( \text{cof}(\mathcal{N}) = \omega_1 \), i.e., the cofinality of the measure ideal \( \mathcal{N} \) is \( \omega_1 \). In particular
- \( \mathfrak{c} > \omega_1 \) and there exists a Boolean algebra \( B \) of cardinality \( \omega_1 \) which is not a union of strictly increasing \( \omega \)-sequence of subalgebras of \( B \).

(C2) There exists a family \( \mathcal{F} \subset [\omega]^\omega \) of cardinality \( \omega_1 \) which is simultaneously independent and splitting. In particular, \( i = s = \omega_1 \).

(C3) There exists a family \( \mathcal{F} \subset [\omega]^\omega \) of cardinality \( \omega_1 \) which is simultaneously maximal almost disjoint and reaping. In particular, \( a = \tau = \omega_1 \).

(C4) \( u = \tau = \omega_1 \), where \( u \) is the smallest cardinality of the base for a non-principal ultrafilter on \( \omega \).

(C5) \( \text{add}(s_0) = \omega_1 \), where \( s_0 \) is the Marczewski’s ideal.

(C6) \( \text{cov}(s_0) = \mathfrak{c} \)

(C7) \( \mathfrak{c} > \omega_1 \) and for every Polish space there exists a partition of \( X \) into \( \omega_1 \) disjoint closed nowhere dense measure zero sets.

Small Sets

(S1) Every perfectly meager set \( S \subset \mathbb{R} \) has cardinality less than \( \mathfrak{c} \).

(S2) Every universally null set \( S \subset \mathbb{R} \) has cardinality less than \( \mathfrak{c} \).

(S3) (Nowik) Every uniformly completely Ramsey null set \( S \subset [\omega]^\omega \) has cardinality less than \( \mathfrak{c} \).
\( \beta \mathbb{N} \) and \( \beta \mathbb{Q} \)

1. There exist \( 2^{\omega_1} \)-many distinct selective ultrafilter on \( \omega \).
2. Every selective filter on \( \omega \) can be extended to a selective ultrafilter.
3. Every selective ultrafilter on \( \omega \) is generated by \( \omega_1 \)-many sets.
4. There exist \( 2^{\omega_1} \)-many distinct non-selective \( P \)-points.
5. There exists a non-principal ultrafilter on \( \mathbb{Q} \) which is crowded, that is, it is generated by (relatively) closed sets without isolated points.

**Other**

- **Total failure of Martin’s Axiom:** \( \check{\tau} > \omega_1 \) and for every non-trivial ccc forcing \( \mathbb{P} \) there exists \( \omega_1 \)-many dense sets in \( \mathbb{P} \) such that no filter intersects all of them.