Covering Property Axiom CPA\textsubscript{cube} and its consequences

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Abstract
In the paper we formulate a Covering Property Axiom CPA\textsubscript{cube}, which holds in the iterated perfect set model, and show that it implies easily the following facts.

(a) For every $S \subset \mathbb{R}$ of cardinality continuum there exists a uniformly continuous function $g: \mathbb{R} \to \mathbb{R}$ with $g[S] = [0, 1]$.

(b) If $S \subset \mathbb{R}$ is either perfectly meager or universally null then $S$ has cardinality less than $\mathfrak{c}$.

(c) $\text{cof}(\mathcal{N}) = \omega_1 < \mathfrak{c}$, i.e., the cofinality of the measure ideal $\mathcal{N}$ is $\omega_1$.

(d) For every uniformly bounded sequence $\langle f_n \in \mathbb{R}^\mathbb{R} \rangle_{n < \omega}$ of Borel functions there are the sequences: $\langle P_\xi \subset \mathbb{R} : \xi < \omega_1 \rangle$ of compact sets and $\langle W_\xi \in [\omega]^\omega : \xi < \omega_1 \rangle$ such that $\mathbb{R} = \bigcup_{\xi < \omega_1} P_\xi$ and for every $\xi < \omega_1$:

\[
\langle f_n \restriction P_\xi \rangle_{n \in W_\xi} \text{ is a monotone uniformly convergent sequence of uniformly continuous functions.}
\]

(e) Total failure of Martin’s Axiom: $\mathfrak{c} > \omega_1$ and for every non-trivial ccc forcing $\mathbb{P}$ there exists $\omega_1$-many dense sets in $\mathbb{P}$ such that no filter intersects all of them.

∗AMS classification numbers: Primary 03E35; Secondary 03E17, 26A03.
Key words and phrases: continuous images, perfectly meager, universally null, cofinality of null ideal, uniform convergence, Martin’s axiom.

†The second author wishes to thank West Virginia University for its hospitality during 1998–2001, where the results presented here were obtained.
1 Axiom CPA$_{\text{cube}}$ and other preliminaries

Our set theoretic terminology is standard and follows that of [4]. In particular, $|X|$ stands for the cardinality of a set $X$ and $\mathfrak{c} = |\mathbb{R}|$. A Cantor set $2^{\omega}$ will be denoted by a symbol $\mathcal{C}$. We use the term Polish space for a complete separable metric space without isolated points.

For a Polish space $X$, the symbol $\text{Perf}(X)$ will stand for the collection of all subsets of $X$ homeomorphic to a Cantor set $\mathcal{C}$. We will consider $\text{Perf}(X)$ as ordered by inclusion. Thus, a family $\mathcal{E} \subset \text{Perf}(X)$ is dense in $\text{Perf}(X)$ provided for every $P \in \text{Perf}(X)$ there exists a $Q \in \mathcal{E}$ such that $Q \subset P$.

Axiom CPA$_{\text{cube}}$ is of the form

$$\mathfrak{c} = \omega^2$$

and if $E \subset \text{Perf}(X)$ is appropriately dense in $\text{Perf}(X)$ then

$$|X \setminus \bigcup E| < \mathfrak{c}$$

for some $E_0 \in [\mathcal{E}]^{\omega_1}$.

If the word “appropriately” in the above is ignored, then it implies the following statement.

Naïve-CPA: If $\mathcal{E}$ is dense in $\text{Perf}(X)$ then $|X \setminus \bigcup E| < \mathfrak{c}$.

It is a very good candidate for our axiom in the sense that it implies all the properties we are interested in. It has, however, one major flaw — it is false!

This is the case since $S \subset X \setminus \bigcup E$ for some dense set $E$ in $\text{Perf}(X)$ provided for each $P \in \text{Perf}(X)$ there is a $Q \in \text{Perf}(X)$ such that $Q \subset P \setminus S$.

This means that the family $\mathcal{G}$ of all sets of the form $X \setminus \bigcup E$, where $E$ is dense in $\text{Perf}(X)$, coincides with the $\sigma$-ideal $s_0$ of Marczewski’s sets, since $\mathcal{G}$ is clearly hereditary. Thus we have

$$s_0 = \left\{ X \setminus \bigcup E : E \text{ is dense in } \text{Perf}(X) \right\}. \quad (1)$$

However, it is well known (see e.g. [17, thm. 5.10]) that there are $s_0$-sets of cardinality $\mathfrak{c}$. Thus, our Naïve-CPA “axiom” cannot be consistent with ZFC.

In order to formulate the real axiom CPA$_{\text{cube}}$ we need the following terminology and notation. A subset $C$ of a product $\mathcal{C}^\eta$ of the Cantor set is said to be a perfect cube if $C = \prod_{n \in \eta} C_n$, where $C_n \in \text{Perf}(\mathcal{C})$ for each $n$. For a fixed Polish space $X$ let $\mathcal{F}_{\text{cube}}$ stand for the family of all continuous injections from a perfect cube $C \subset \mathcal{C}^\omega$ onto a set $P$ from $\text{Perf}(X)$. We consider each function $f \in \mathcal{F}_{\text{cube}}$ from $C$ onto $P$ as a coordinate system imposed on $P$. We say that $P \in \text{Perf}(X)$ is a cube if it is determined by an (implicitly given) witness function $f \in \mathcal{F}_{\text{cube}}$ onto $P$, and $Q$ is a subcube of a cube $P \in \text{Perf}(X)$ provided $Q = f[C]$, where $f \in \mathcal{F}_{\text{cube}}$ is a witness function for $P$ and $C \subset \text{dom}(f) \subset \mathcal{C}^\omega$ is a perfect cube. Here and in what follows the symbol $\text{dom}(f)$ stands for the domain of $f$.

We say that a family $\mathcal{E} \subset \text{Perf}(X)$ is cube dense in $\text{Perf}(X)$ provided every cube $P \in \text{Perf}(X)$ contains a subcube $Q \in \mathcal{E}$. More formally, $\mathcal{E} \subset \text{Perf}(X)$ is cube dense provided

$$\forall f \in \mathcal{F}_{\text{cube}} \exists g \in \mathcal{F}_{\text{cube}} (g \subset f \& \text{range}(g) \in \mathcal{E}). \quad (2)$$
It is easy to see that the notion of cube density is a generalization of the notion of density as defined in the first paragraph of this section:

if $E$ is cube dense in $\text{Perf}(X)$ then $E$ is dense in $\text{Perf}(X)$. \hfill (3)

On the other hand, the converse implication is not true, as shown by the following simple example.

Example 1.1 Let $X = \mathcal{C} \times \mathcal{C}$ and let $E$ be the family of all $P \in \text{Perf}(X)$ such that either

- all vertical sections $P_x = \{y \in \mathcal{C} : (x, y) \in P\}$ of $P$ are countable, or
- all horizontal sections $P^y = \{x \in \mathcal{C} : (x, y) \in P\}$ of $P$ are countable.

Then $E$ is dense in $\text{Perf}(X)$, but it is not cube dense in $\text{Perf}(X)$.

Proof. To see that $E$ is dense in $\text{Perf}(X)$ let $R \in \text{Perf}(X)$. We need to find a $P \subset R$ with $P \in E$. Clearly at least one of the projections $\pi_0(R)$ or $\pi_1(R)$ is uncountable. Assume that $\pi_0(R)$ is uncountable and let $p : \pi_0(R) \to \mathcal{C}$ be a Borel function. (For example, if $p$ is defined by $p(x) = \min R_x$ then $p \subset R$ is Baire class 1.) So, there is a $Q \in \text{Perf}(\mathcal{C})$ such that $p \upharpoonright Q$ is continuous. In particular, $p \upharpoonright Q$ (identified with its graph) is a closed subset of $X = \mathcal{C} \times \mathcal{C}$. So $P = p \upharpoonright Q \in E$ is a subset of $R$.

To see that $E$ is not $\mathcal{F}_{\text{cube}}$-dense in $\text{Perf}(X)$ it is enough to notice that $P = X = \mathcal{C} \times \mathcal{C}$ considered as a cube, where the second coordinate is identified with $\mathcal{C}^{\omega \setminus \{0\}}$, has no subcube in $E$. More formally, let $h$ be a homeomorphism from $\mathcal{C}$ onto $\mathcal{C}^{\omega \setminus \{0\}}$, let $g : \mathcal{C} \times \mathcal{C} \to \mathcal{C}^\omega = \mathcal{C} \times \mathcal{C}^{\omega \setminus \{0\}}$ be given by $g(x, y) = (x, h(y))$, and let $f = g^{-1} : \mathcal{C}^\omega \to \mathcal{C} \times \mathcal{C}$ be the coordinate function making $\mathcal{C} \times \mathcal{C} = \text{range}(f)$ a cube. Then $\text{range}(f)$ does not contain a subcube from $E$. \hfill $\blacksquare$

With these notions in hand we are ready to formulate our axiom CPA_{cube}.

CPA_{cube}: $\kappa = \omega_2$ and for every Polish space $X$ and every cube dense family $E \subset \text{Perf}(X)$ there is an $E_0 \subset E$ such that $|E_0| \leq \omega_1$ and $|X \setminus \bigcup E_0| \leq \omega_1$.

The proof that CPA_{cube} holds in the iterated perfect set model can be found in [6] and [7].

It is also worth noticing that in order to check that $E$ is cube dense it is enough to consider in condition (2) only functions $f$ defined on the entire space $\mathcal{C}^\omega$, that is

Fact 1.2 $E \subset \text{Perf}(X)$ is cube dense if and only if

$$\forall f \in \mathcal{F}_{\text{cube}}, \text{ dom}(f) = \mathcal{C}^\omega, \exists g \in \mathcal{F}_{\text{cube}} \ (g \subset f \& \text{range}(g) \in E). \hfill (4)$$

Proof. To see this, let $\Phi$ be the family of all bijections $h = \langle h_n \rangle_{n < \omega}$ between perfect subcubes $\prod_{n \in \omega} D_n$ and $\prod_{n \in \omega} C_n$ of $\mathcal{C}^\omega$ such that each $h_n$ is a homeomorphism between $D_n$ and $C_n$. Then

$$f \circ h \in \mathcal{F}_{\text{cube}} \ \text{for every} \ f \in \mathcal{F}_{\text{cube}} \ \text{and} \ h \in \Phi \ \text{with range}(h) \subset \text{dom}(f). \hfill (5)$$
Now take an arbitrary $f: C \rightarrow X$ from $\mathcal{F}_{\text{cube}}$ and choose an $h \in \Phi$ mapping $\mathcal{C}^\omega$ onto $C$. Then $\hat{f} = f \circ h \in \mathcal{F}_{\text{cube}}$ maps $\mathcal{C}^\omega$ into $X$ and, using (4), we can find $\hat{g} \in \mathcal{F}_{\text{cube}}$ such that $\hat{g} \subset \hat{f}$ and $\text{range}(\hat{g}) \in \mathcal{E}$. Then $g = f \mid h[\text{dom}(\hat{g})]$ satisfies condition (2).

Next, let us consider

$$s_0^{\text{cube}} = \left\{ X \setminus \bigcup_{(E, \mathcal{E})} \mathcal{E} : \mathcal{E} \text{ is cube dense in } \text{Perf}(X) \right\}$$

(6)

$$s_0^{\text{cube}} = \left\{ S \subset X : \forall \text{ cube } P \in \text{Perf}(X) \exists \text{ subcube } Q \subset P \setminus S \right\}.$$

Notice that

**Fact 1.3** $[X]^{<\kappa} \subset s_0^{\text{cube}} \subset s_0$ for every Polish space $X$.

An easy proof of this Fact 1.3 can be found in [7]. It can be also shown in ZFC that $s_0^{\text{cube}}$ forms a $\sigma$-ideal. However, neither of these facts will be used in the sequel. On the other hand we will be interested in the following proposition.

**Proposition 1.4** If $\text{CPA}_{\text{cube}}$ holds then $s_0^{\text{cube}} = [X]^{<\omega_1}$.

**Proof.** It is obvious that $\text{CPA}_{\text{cube}}$ implies $s_0^{\text{cube}} \subset [X]^{<\kappa}$. We will not be interested in the other inclusion, though it follows immediately from Fact 1.3.

**Remark 1.5** $s_0^{\text{cube}} \neq [X]^{<\omega_1}$ in a model obtained by adding Sacks numbers side-by-side. In particular $\text{CPA}_{\text{cube}}$ is false in this model.

**Proof.** This follows from the fact that $s_0^{\text{cube}} = [X]^{<\omega_1}$ implies the property (A) (see Corollary 2.2) while it is false in the model mentioned above, as noticed by Miller in [16, p. 581]. (In this model the set $X$ of all Sacks generic numbers cannot be mapped continuously onto $[0, 1]$.)

## 2 Continuous images of sets of cardinality $\mathfrak{c}$

An important quality of the ideal $s_0^{\text{cube}}$, and so the power of the assumption $s_0^{\text{cube}} = [X]^{<\kappa}$, is well depicted by the following fact.

**Proposition 2.1** If $X$ is a Polish space and $S \subset X$ does not belong to $s_0^{\text{cube}}$ then there exist a $T \in [S]^{<\kappa}$ and a uniformly continuous function $h$ from $T$ onto $\mathcal{C}$.

**Proof.** Take an $S$ as above and let $f: \mathcal{C}^\omega \rightarrow X$ be a continuous injection such that $f(C) \cap S \neq \emptyset$ for every perfect cube $C$. Let $g: \mathcal{C} \rightarrow \mathcal{C}$ be a continuous function such that $g^{-1}(y)$ is perfect for every $y \in \mathcal{C}$. Then $h_0 = g \circ \pi_0 \circ f^{-1}: f[\mathcal{C}^\omega] \rightarrow \mathcal{C}$ is uniformly continuous. Moreover, if $T = S \cap f[\mathcal{C}^\omega]$ then $h_0[T] = \mathcal{C}$ since

$$T \cap h_0^{-1}(y) = T \cap f[\pi_0^{-1}(g^{-1}(y))] = S \cap f[g^{-1}(y) \times \mathcal{C} \times \mathcal{C} \times \cdots] \neq \emptyset$$

for every $y \in \mathcal{C}$.
Corollary 2.2 Assume $s_0^\text{cube} = [X]^{<\tau}$ for a Polish space $X$. If $S \subset X$ has cardinality $\tau$ then there exists a uniformly continuous function $f : X \to [0, 1]$ such that $f[S] = [0, 1]$.

In particular, CPA$_{\text{cube}}$ implies property (a).

Proof. If $S$ is as above then, by CPA$_{\text{cube}}, S \notin s_0^\text{cube}$. Thus, by Proposition 2.1 there exists a uniformly continuous function $h$ from a subset of $S$ onto $C$. Consider $C$ as a subset of $[0, 1]$ and let $\hat{h} : X \to [0, 1]$ be a uniformly continuous extension of $h$. If $g : [0, 1] \to [0, 1]$ is continuous and such that $g[C] = [0, 1]$ then $f = g \circ \hat{h}$ is as desired.

The fact that (a) holds in the iterated perfect set model was first proved by A. Miller in [16].

It is worth to note here that the function $f$ in Corollary 2.2 cannot be required to be either monotone or in the class “$D^1$” of all functions having finite or infinite derivative at every point. This follows immediately from the following proposition, since each function which is either monotone or “$D^1$” belongs to the Banach class

$$(T_2) = \{f \in C(\mathbb{R}) : \{y \in \mathbb{R} : |f^{-1}(y)| > \omega \} \in \mathcal{N}\}.$$

(See [10] or [19, p. 278].)

Proposition 2.3 There exists, in ZFC, an $S \in [\mathbb{R}]^\tau$ such that $[0, 1] \notin f[S]$ for every $f \in (T_2)$.

Proof. Let $\{f_\xi : \xi < \tau\}$ be an enumeration of all functions from $(T_2)$ whose range contains $[0, 1]$. Construct by induction a sequence $\langle (s_\xi, y_\xi) : \xi < \tau \rangle$ such that for every $\xi < \tau$

(i) $y_\xi \in [0, 1] \setminus f_\xi[\{s_\zeta : \zeta < \xi\}]$ and $|f_\xi^{-1}(y_\xi)| \leq \omega$.

(ii) $s_\xi \in \mathbb{R} \setminus \left(\{s_\zeta : \zeta < \xi\} \cup \bigcup_{\zeta \leq \xi} f_\zeta^{-1}(y_\zeta)\right)$.

Then the set $S = \{s_\xi : \xi < \tau\}$ is as required since $y_\xi \in [0, 1] \setminus f_\xi[S]$ for every $\xi < \tau$.

3 Perfectly meager and universally null sets

The fact that (b) holds in the iterated perfect set model was first proved by A. Miller in [16].

Theorem 3.1 If $S \subset \mathbb{R}$ is either perfectly meager or universally null then $S \in s_0^\text{cube}$.

In particular, CPA$_{\text{cube}}$ implies property (b).
Proof. Take an $S \subset \mathbb{R}$ which is either perfectly meager or universally null and let $f : \mathcal{C}^\omega \to \mathbb{R}$ be a continuous injection. Then $S \cap f[\mathcal{C}^\omega]$ is either meager or null in $f[\mathcal{C}^\omega]$. Thus $G = \mathcal{C}^\omega \setminus f^{-1}(S)$ is either comeager or of full measure in $\mathcal{C}^\omega$. Hence the theorem follows immediately from the following claim.

Claim 3.2 Consider $\mathcal{C}^\omega$ with standard topology and standard product measure. If $G$ is a Borel subset of $\mathcal{C}^\omega$ which is either of second category or of positive measure then $G$ contains a perfect cube $\prod_{i<\omega} P_i$.

The measure version of the claim is a variant the following theorem:

(m) For every full measure subset $H$ of $[0, 1] \times [0, 1]$ there are a perfect set $P \subset [0, 1]$ and a positive inner measure subset $\hat{H}$ of $[0, 1]$ such that $P \times \hat{H} \subset H$.

which was proved by Eggleston [9] and, independently, by Brodskii [3]. The category version of the claim is a consequence of the category version of (m):

(c) For every Polish space $X$ and every comeager subset $G$ of $X \times X$ there are a perfect set $P \subset X$ and a comeager subset $\hat{G}$ of $X$ such that $P \times \hat{G} \subset G$.

This well known result can be found in [12, Exercise 19.3]. (Its version for $\mathbb{R}^2$ is also proved, for example, in [8, condition (⋆), p. 416].) For completeness, we will show here in detail how to deduce the claim from (m) and (c).

We will start the argument with a simple fact, in which we will use the following notations. If $X$ is a Polish space endowed with a Borel measure then $\psi_0(X)$ will stand for the sentence

$$\psi_0(X) : \text{For every full measure subset } H \text{ of } X \times X \text{ there are a perfect set } P \subset X \text{ and a positive inner measure subset } \hat{H} \text{ of } X \text{ such that } P \times \hat{H} \subset H.$$ 

Thus $\psi_0([0, 1])$ is a restatement of (m). We will also use the following seemingly stronger variants of $\psi_0(X)$.

$$\psi_1(X) : \text{For every full measure subset } H \text{ of } X \times X \text{ there are a perfect set } P \subset X \text{ and a subset } \hat{H} \text{ of } X \text{ of full measure such that } P \times \hat{H} \subset H.$$ 

$$\psi_2(X) : \text{For a subset } H \text{ of } X \times X \text{ of positive inner measure there are a perfect set } P \subset X \text{ and a positive inner measure subset } \hat{H} \text{ of } X \text{ such that } P \times \hat{H} \subset H.$$ 

Fact 3.3 Let $n = 1, 2, 3, \ldots$.

(i) If $E \subset \mathbb{R}^n$ has a positive Lebesgue measure then $\mathbb{Q}^n + E = \bigcup_{q \in \mathbb{Q}^n} (q + E)$ has a full measure.

(ii) $\psi_k(X)$ holds for all $k < 3$ and $X \in \{[0, 1), (0, 1), \mathbb{R}, \mathcal{C}\}$.

Proof. (i) Let $\lambda$ be the Lebesgue measure on $\mathbb{R}^n$ and for $\varepsilon > 0$ and $x \in \mathbb{R}^n$ let $B(x, \varepsilon)$ be an open ball in $\mathbb{R}^n$ of radius $\varepsilon$ centered at $x$. By way of contradiction assume that there exists a positive measure set $A \subset \mathbb{R}^n$ disjoint with $\mathbb{Q}^n + E$. Let $a \in A$ and $x \in E$ be the Lebesgue density points of $A$ and $X$, respectively. Take
C preserves product measure, we can identify $\psi$ on $\hat{H}$ relatively open subset of $\hat{H}$ necessary, we can assume that $\hat{A}$ is covered by countably many compact sets ($\hat{A} \cap (q + E) \neq \emptyset$ since $B(a, \varepsilon/2) \subset B(a, \varepsilon) \cap B(q + x, \varepsilon)$ and thus $\lambda(A \cap (q + E) \cap B(a, \varepsilon/2)) > \lambda(B(a, \varepsilon/2)) - 2 \cdot 4^{-n}\lambda(B(a, \varepsilon)) \geq 0$. Hence $A \cap (\mathbb{Q}^n + E) \neq \emptyset$ contradicting the choice of $A$.

(ii) First note that $\psi_k(\mathbb{R}) \Leftrightarrow \psi_k((0, 1)) \Leftrightarrow \psi_k([0, 1]) \Leftrightarrow \psi_k(\mathbb{E})$ for every $k < 3$. This is justified by the fact that for the mappings $f: (0, 1) \rightarrow \mathbb{R}$ given by $f(x) = \cot(x\pi)$, the identity mapping $id: (0, 1) \rightarrow [0, 1]$, and $d: \mathbb{E} \rightarrow [0, 1]$ given by $d(x) = \sum_{i<\omega} \frac{x(i)}{2^i}$, the image and the preimage of measure zero set (full measure set) is of measure zero (full measure).

Since, by (m), $\psi_0([0, 1])$ is true, we have also that $\psi_0(X)$ holds also for $X \in \{(0, 1), \mathbb{R}, \mathbb{E}\}$. To finish the proof it is enough to show that $\psi_0(\mathbb{R})$ implies $\psi_1(\mathbb{R})$ and $\psi_2(\mathbb{R})$.

To prove $\psi_1(\mathbb{R})$ let $H$ be a full measure subset of $\mathbb{R} \times \mathbb{R}$ and let us define $H_0 = \bigcap_{q \in \mathbb{Q}} ((0, q) \cup H)$. Then $H_0$ is still of full measure so, by $\psi_0(\mathbb{R})$, there are perfect set $P \subset \mathbb{R}$ and a positive inner measure subset $\hat{H}_0$ of $\mathbb{R}$ such that $P \times \hat{H}_0 \subset H_0$. Thus, for every $q \in \mathbb{Q}$ we also have $P \times (q + \hat{H}_0) \subset (0, q) + H_0 = H_0$. Let $\hat{H} = \bigcup_{q \in \mathbb{Q}} (q + \hat{H}_0)$. Then $P \times \hat{H} \subset H_0 \subset H$ and, by (i), $\hat{H}$ has full measure. So, $\psi_1(\mathbb{R})$ is proved.

To prove $\psi_2(\mathbb{R})$ let $H \subset \mathbb{R} \times \mathbb{R}$ be of positive inner measure. Decreasing $H$, if necessary, we can assume that $H$ is compact. Let $H_0 = \mathbb{Q}^2 + H$. Then, by (i), $H_0$ is of full measure so, by $\psi_0(\mathbb{R})$, there are perfect set $P_0 \subset \mathbb{R}$ and a positive inner measure subset $\hat{H}_0$ of $\mathbb{R}$ such that $P_0 \times H_0 \subset H_0$. Once again, decreasing $P_0$ and $\hat{H}_0$ if necessary, we can assume that they are homeomorphic to $\mathbb{E}$ and that no relatively open subset of $\hat{H}_0$ has measure zero. Since $P_0 \times H_0 \subset \bigcup_{q \in \mathbb{Q}} (q + H)$ is covered by countably many compact sets $(P_0 \times \hat{H}_0) \cap (q + H)$ with $q \in \mathbb{Q}$, there is a $q = \langle q_0, q_1 \rangle \in \mathbb{Q}^2$ such that $(P_0 \times \hat{H}_0) \cap (q + H)$ has a non-empty interior in $P_0 \times \hat{H}_0$. Let $U$ and $V$ be non-empty clopen subsets of $P_0$ and $\hat{H}_0$, respectively, such that $U \times V \subset (P_0 \times \hat{H}_0) \cap (q + H) \subset \langle q_0, q_1 \rangle + H$. Then $U$ and $V$ are perfect and $V$ has positive measure. Let $P = -q_0 + U$ and $\hat{H} = -q_1 + V$. Then $P \times \hat{H} = (-q_0 + U) \times (-q_1 + V) = -\langle q_0, q_1 \rangle + (U \times V) \subset H$, so $\psi_2(\mathbb{R})$ holds.

**Proof of Claim 3.2.** Since the natural homeomorphism between $\mathbb{E}$ and $\mathbb{E} \setminus \{0\}$ preserves product measure, we can identify $\mathbb{E} \times \mathbb{E} \setminus \{0\}$ with $\mathbb{E} \times \mathbb{E}$ considered with its usual topology and its usual product measure. With this identification the result follows easily, by induction on coordinates, from the following fact.

(•) For every Borel subset $H$ of $\mathbb{E} \times \mathbb{E}$ which is of second category (of positive measure) there are a perfect set $P \subset \mathbb{E}$ and a second category (positive measure) subset $\hat{H}$ of $\mathbb{E}$ such that $P \times \hat{H} \subset H$.

The measure version of (•) is a restatement of $\psi_2(\mathbb{E})$, which was proved in Fact 3.3(ii). To see the category version of (•) let $H$ be a Borel subset of $\mathbb{E} \times \mathbb{E}$ of second category. Then there are clopen subsets $U$ and $V$ of $\mathbb{E}$ such that $H_0 = H \cap (U \times V)$ is comeager in $U \times V$. Since $U$ and $V$ are homeomorphic to
\( \mathfrak{c}, \) we can apply (c) to \( H_0 \) and \( U \times V \) we can find a perfect set \( P \subset U \) and a comeager Borel subset \( \bar{H} \) of \( V \) such that \( P \times \bar{H} \subset H_0 \subset H \), finishing the proof.

\[ \square \]

4 \( \text{cof(} \mathcal{N} \text{)} = \omega_1 < \mathfrak{c} \)

Next we show that CPA_{cube} implies that \( \text{cof(} \mathcal{N} \text{)} = \omega_1 \). So, under CPA_{cube}, all cardinals from Cichoń’s diagram (see e.g. [1]) are equal to \( \omega_1 \). The fact that this holds in the iterated perfect set model has been well known.

Let \( C_H \) be the family of all subsets \( \prod_{n<\omega} T_n \) of \( \omega^\omega \) such that \( T_n \in [\omega]^{\leq n+1} \) for all \( n < \omega \). We will use the following characterization.

**Proposition 4.1** (Bartoszyński [1, thm. 2.3.9])

\[
\text{cof(} \mathcal{N} \text{)} = \min \left\{ |\mathcal{F}| : \mathcal{F} \subset C_H \land \bigcup \mathcal{F} = \omega^\omega \right\}.
\]

**Lemma 4.2** The family \( C_H^* = \{ X \subset \omega^\omega : X \subset T \text{ for some } T \in C_H \} \) is \( \mathcal{F}_{\text{cube}} \)-dense in \( \text{Perf}(\omega^\omega) \).

**Proof.** Let \( f: \mathcal{C}^\omega \rightarrow \omega^\omega \) be a continuous function. By (4) it is enough to find a perfect cube \( C \) in \( \mathcal{C}^\omega \) such that \( f[C] \in C_H^* \).

Construct, by induction on \( n < \omega \), the families \( \{ E_n^i : s \in 2^n \land i < \omega \} \) of non-empty clopen subsets of \( \mathcal{C} \) such that for every \( n < \omega \) and \( s, t \in 2^n \)

(i) \( E_s^i = E_t^i \) for every \( n \leq i < \omega \);

(ii) \( E_s^i \) and \( E_t^{i+1} \) are disjoint subsets of \( E_s^i \) for every \( i < n+1 \);

(iii) for every \( \langle s_i \rangle \in 2^n : i < \omega \)

\[
f(x_0) \upharpoonright 2^{(n+1)^2} = f(x_1) \upharpoonright 2^{(n+1)^2} \text{ for every } x_0, x_1 \in \prod_{i<\omega} E_{s_i}.
\]

For each \( i < \omega \) the fusion of \( \{ E_n^i : s \in 2^{<\omega} \} \) will give us the \( i \)-th coordinate set of the desired perfect cube \( C \).

Condition (iii) can be ensured by uniform continuity of \( f \). Indeed, let \( \delta > 0 \) be such that \( f(x_0) \upharpoonright 2^{(n+1)^2} = f(x_1) \upharpoonright 2^{(n+1)^2} \) for every \( x_0, x_1 \in \mathcal{C}^\omega \) of distance less than \( \delta \). Then it is enough to choose \( \{ E_n^i : s \in 2^n \land i < \omega \} \) such that (i) and (ii) are satisfied and every set \( \prod_{i<\omega} E_{s_i} \) from (iii) has diameter less than \( \delta \). This finishes the construction.

Next for every \( i, n < \omega \) let \( E_n^i = \bigcup \{ E_s^i : s \in 2^n \} \) and \( E_n = \prod_{i<\omega} E_n^i \). Then \( C = \bigcap_{n<\omega} E_n = \prod_{i<\omega} (\bigcap_{n<\omega} E_n^i) \) is a perfect cube, since \( \bigcap_{n<\omega} E_n^i \in \text{Perf}(\mathcal{C}) \) for every \( i < \omega \). Thus, to finish the proof it is enough to show that \( f[C] \in C_H^* \).

So, for every \( k < \omega \) let \( n < \omega \) be such that \( 2^{n^2} \leq k + 1 < 2^{(n+1)^2} \), put

\[
T_k = \{ f(x)(k) : x \in E_n \} = \left\{ f(x)(k) : x \in \prod_{i<\omega} E_{s_i} \text{ for some } \langle s_i \rangle \in 2^n : i < \omega \right\},
\]

\[ \square \]
and notice that $T_k$ has at most $2^{n^2} \leq k + 1$ elements. Indeed, by (iii), the set \( \{ f(x)(k): x \in \prod_{i<\omega} E_s_i \} \) has precisely one element for every \( \langle s_i \in 2^n: i < \omega \rangle \) while (i) implies that \( \{ \prod_{i<\omega} E_s_i: \langle s_i \in 2^n: i < \omega \rangle \} \) has $2^{n^2}$ elements. Therefore \( \prod_{k<\omega} T_k \in C_H \).

To finish the proof it is enough to notice that \( f[C] \subset \prod_{k<\omega} T_k \).

**Corollary 4.3** If CPA$\uparrow$ holds then \( \text{cof}(N) = \omega_1 \).

**Proof.** By CPA$\uparrow$ and Lemma 4.2 there exists an \( F \in \mathcal{C}_H \leq \omega_1 \) such that \( |\omega | \setminus \bigcup F | \leq \omega_1 \). This and Proposition 4.1 imply \( \text{cof}(N) = \omega_1 \).

## 5 Pointwise convergent of subsequences of real-valued functions

A sequence \( \langle f_n \rangle_{n<\omega} \) of real-valued functions is *uniformly bounded* provided there exists an \( r \in \mathbb{R} \) such that \( \text{range}(f_n) \subset [-r, r] \) for every \( n \). In 1932 Mazurkiewicz [15] proved the following variant of Egorov’s theorem.

*For every uniformly bounded sequence \( \langle f_n \rangle_{n<\omega} \) of real-valued continuous functions defined on a Polish space \( X \) there exists a subsequence which is uniformly convergent on some perfect set \( P \).*

The main result of this section is the following theorem.

**Theorem 5.1** If CPA$\uparrow$ holds then

\( \langle f_n \rangle_{n<\omega} \) is monotone and uniformly convergent sequence of uniformly continuous functions.

Theorem 5.1 is a variant of [5, theorem 2] and its corollary, according to which condition \( \text{(\ast)} \) for continuous functions \( f_n \) can be deduced from the assumptions that \( \text{cof}(N) = \omega_1 \) and there exists a selective \( \omega_1 \)-generated ultrafilter on \( \omega \).

**Proof.** We first note that the family \( \mathcal{E} \) of all \( P \in \text{Perf}(X) \) for which there exists a \( W \in [\omega]^\omega \) such that

\[ P_n | P \in W \] is a monotone and uniformly convergent sequence of uniformly continuous functions.

Indeed, let \( g \in \mathcal{F}_\text{cube} \), \( g: \mathcal{C}^\omega \rightarrow X \), and consider the functions \( h_n = f_n \circ g \). Since \( h = \{ h_n: n < \omega \}: \mathcal{C}^\omega \rightarrow \mathbb{R}^\omega \) is Borel measurable, there is a dense \( G_\delta \)
subset $G$ of $\mathcal{C}$ such that $h \upharpoonright G$ is continuous. So, we can find a perfect cube $C \subset G \subset \mathcal{C}$, and for this $C$ function $h \upharpoonright C$ is continuous. Thus, identifying the coordinate spaces of $C$ with $\mathcal{C}$, without loss of generality we can assume that $C = \mathcal{C}$, that is, that each function $h_n: \mathcal{C} \to \mathbb{R}$ is continuous. Now, by [21, thm. 6.9], there is a perfect cube $C$ in $\mathcal{C}$ and a $W \in [\omega]^{\omega}$ such that the sequence $\langle h_n \upharpoonright C \rangle_{n \in W}$ is monotone and uniformly convergent. So $P = g[C]$ is in $\mathcal{E}$.

Now, by CPA$_\text{cube}$, there exists an $\mathcal{E}_0 \in [\mathcal{E}]^{\leq \omega_1}$ such that $|X \setminus \bigcup \mathcal{E}_0| \leq \omega_1$. Then $\{P_\xi: \xi < \omega_1\} = \mathcal{E}_0 \cup \{x: x \in X \setminus \bigcup \mathcal{E}_0\}$ is as desired: if $P_\xi \in \mathcal{E}_0$ then the existence of an appropriate $W_\xi$ follows from the definition of $\mathcal{E}$. If $P_\xi$ is a singleton, then the existence of $W_\xi$ follows from the fact that every sequence of reals has a monotone subsequence.

6 Total failure of Martin’s Axiom

In this section we prove that CPA$_\text{cube}$ implies the total failure of Martin’s Axiom, that is, the property that

for every non-trivial ccc forcing $\mathbb{P}$ there exists $\omega_1$-many dense sets in $\mathbb{P}$ such that no filter intersects all of them.

The consistency of this fact with $\mathfrak{c} > \omega_1$ was first proved by Baumgartner [2] in a model obtained by adding Sacks reals side-by-side. The topological and boolean algebraic formulations of the theorem follow immediately from the following proposition.

Proposition 6.1 The following conditions are equivalent.

(a) For every non-trivial ccc forcing $\mathbb{P}$ there exists $\omega_1$-many dense sets in $\mathbb{P}$ such that no filter intersects all of them.

(b) Every compact ccc topological space without isolated points is a union of $\omega_1$ nowhere dense sets.

(c) For every atomless ccc complete Boolean algebra $B$ there exists $\omega_1$-many dense sets in $B$ such that no filter intersects all of them.

(d) For every atomless ccc complete Boolean algebra $B$ there exists $\omega_1$-many maximal antichains in $B$ such that no filter intersects all of them.

(e) For every countably generated atomless ccc complete Boolean algebra $B$ there exists $\omega_1$-many maximal antichains in $B$ such that no filter intersects all of them.

1Actually [21, thm. 6.9] is stated for functions defined on $[0, 1]^{\omega}$. However, the proof presented there for works also for functions defined on $\mathcal{C}$.
Proof. The equivalence of the conditions (a), (b), (c), and (d) is well known. In particular, equivalence (a)–(c) is explicitly given in [2, thm. 0.1]. Clearly (d) implies (e). The remaining implication, (e)⇒(d), is a version of the theorem from [14, p. 158]. However, it is expressed there in a bit different language, so we include here its proof.

So, let \( (B, \lor, \land, 0, 1) \) be an atomless ccc complete Boolean algebra. For every \( \sigma \in 2^{<\omega_1} \) define, by induction on the length \( \text{dom}(\sigma) \) of a sequence \( \sigma \), a \( b_\sigma \in B \) such that the following conditions are satisfied.

- \( b_0 = 1 \).
- \( b_\sigma \) is a disjoint union of \( b_{\sigma \cdot 0} \) and \( b_{\sigma \cdot 1} \).
- If \( b_\sigma > 0 \) then \( b_{\sigma \cdot 0} > 0 \) and \( b_{\sigma \cdot 1} > 0 \).
- If \( \lambda = \text{dom}(\sigma) \) is a limit ordinal then \( b_\sigma = \bigwedge_{\xi < \lambda} b_{\sigma | \xi} \).

Let \( T = \{ s \in 2^{<\omega_1} : b_s > 0 \} \). Then \( T \) is a subtree of \( 2^{<\omega_1} \); its levels determine antichains in \( B \), so they are countable.

First assume that \( T \) has a countable height. Then \( T \) itself is countable. Let \( B_0 \) be the smallest complete subalgebra of \( B \) containing \( \{ b_\sigma : \sigma \in T \} \) and notice that \( B_0 \) is atomless. Indeed, if there were an atom \( a \) in \( B_0 \) then \( S = \{ \sigma \in T : a \leq b_\sigma \} \) would be a branch in \( T \) so that \( \delta = \bigcup S \) would belong to \( 2^{<\omega_1} \). Since \( b_\delta \geq a > 0 \), we would also have \( \delta \in T \). But then \( a \leq b_\delta = b_{\delta \cdot 0} \lor b_{\delta \cdot 1} \) so that either \( \delta \cdot 0 \) or \( \delta \cdot 1 \) belongs to \( S \), which is impossible.

Thus, \( B_0 \) is a complete, countably generated, atomless subalgebra of \( B \). So, by (e), there exists a family \( \mathcal{A} \) of \( \omega_1 \)-many maximal antichains in \( B_0 \) with no filter in \( B_0 \) intersecting all of them. But then each \( A \subset \mathcal{A} \) is also a maximal antichain in \( B \) and no filter in \( B \) would intersect all of them. So, we have (d).

Next, assume that \( T \) has height \( \omega_1 \) and for every \( \alpha < \omega_1 \) let

\[ T_\alpha = \{ \sigma \in T : \text{dom}(\sigma) = \alpha \} \]

be the \( \alpha \)-th level of \( T \). Also let \( b_\alpha = \bigvee_{\sigma \in T_\alpha} b_\sigma \). Notice that \( b_\alpha = b_{\alpha + 1} \) for every \( \alpha < \omega_1 \). On the other hand, it may happen that \( b_\lambda > \bigwedge_{\alpha < \lambda} b_\alpha \) for some limit \( \lambda < \omega_1 \); however, this may happen only countably many times, since \( B \) is ccc. Thus, there is an \( \alpha < \omega_1 \) such that \( b_\beta = b_\alpha \) for every \( \alpha < \beta < \omega_1 \).

Now, let \( B_0 \) be the smallest complete subalgebra of \( B \) below \( 1 \setminus b_\alpha \) containing \( \{ b_\sigma : b_\sigma \setminus b_\alpha, \sigma \in T \} \). Then \( B_0 \) is countably generated and, as before, it can be shown that \( B_0 \) is atomless. Thus, there exists a family \( \mathcal{A}_0 \) of \( \omega_1 \)-many maximal antichains in \( B_0 \) with no filter in \( B_0 \) intersecting all of them. Then no filter in \( B \) containing \( 1 \setminus b_\alpha \) intersects every \( A \in \mathcal{A}_0 \). But for every \( \alpha < \beta < \omega_1 \) the set \( A_\beta = \{ b_\sigma : \sigma \in T_\beta \} \) is a maximal antichain in \( B \) below \( b_\alpha \). Therefore, \( \mathcal{A}_1 = \{ A_\beta : \alpha < \beta < \omega_1 \} \) is an uncountable family of maximal antichains in \( B \) below \( b_\alpha \) with no filter in \( B \) containing \( b_\alpha \) intersecting every \( A \in \mathcal{A}_1 \). Then it is easy to see that the family \( \mathcal{A} = \{ A_0 \cup A_1 : a_0 \in \mathcal{A}_0 \land A_1 \in \mathcal{A}_1 \} \) is a family of \( \omega_1 \)-many maximal antichains in \( B \) with no filter in \( B \) intersecting all of them.

This proves (d).
Theorem 6.2 \( \text{CPA}_{\text{cube}} \) implies the total failure of Martin’s Axiom.

Proof. Let \( \mathcal{A} \) be a countably generated atomless ccc complete Boolean algebra and let \( \{ A_n : n < \omega \} \) generate \( \mathcal{A} \). By Proposition 6.1 it is enough to show that \( \mathcal{A} \) contains \( \omega_1 \)-many maximal antichains such that no filter in \( \mathcal{A} \) intersects all of them.

Next let \( \mathcal{B} \) be the \( \sigma \)-algebra of Borel subsets of \( \mathcal{C} = 2^n \). By Proposition 6.1, it is enough to show that \( \mathcal{A} \) contains \( \omega_1 \)-many maximal antichains such that no filter in \( \mathcal{A} \) intersects all of them.

Next let \( B \) be the \( \sigma \)-algebra of Borel subsets of \( C = 2^n \). Recall that it is a free countably generated \( \sigma \)-algebra, with free generators \( B_i = \{ s \in C : s(i) = 0 \} \).

Define \( h_0 : \{ B_n : n < \omega \} \rightarrow \{ A_n : n < \omega \} \) by \( h_0(B_n) = A_n \) for all \( n < \omega \). Then \( h_0 \) can be uniquely extended to a \( \sigma \)-homomorphism \( h : \mathcal{B} \rightarrow \mathcal{A} \) between \( \sigma \)-algebras \( \mathcal{B} \) and \( \mathcal{A} \). (See e.g. [20, 34.1 p. 117].) Let \( I = \{ B \in \mathcal{B} : h[B] = 0 \} \). Then \( I \) is a \( \sigma \)-ideal in \( \mathcal{B} \) and the quotient algebra \( \mathcal{B}/I \) is isomorphic to \( \mathcal{A} \). (Compare also Loomis-Sikorski theorem in [20, p. 117] or [13].) In particular, \( I \) contains all singletons and is ccc, since \( \mathcal{A} \) is atomless and ccc.

It follows that we need only to consider complete Boolean algebras of the form \( \mathcal{B}/I \), where \( I \) is some ccc \( \sigma \)-ideal of Borel sets containing all singletons. To prove that such an algebra has \( \omega_1 \) maximal antichains as desired, it is enough to prove that

\[ (\ast) \mathcal{C} \text{ is a union of } \omega_1 \text{ perfect sets } \{ N_\xi : \xi < \omega_1 \} \text{ which belong to } I. \]

Indeed, assume that \( (\ast) \) holds and for every \( \xi < \omega_1 \) let \( D_\xi \) be a family of all \( B \in \mathcal{B} \setminus I \) with closures \( \text{cl}(B) \) disjoint from \( N_\xi \). Then \( D_\xi = \{ B/I : B \in D_\xi \} \) is dense in \( \mathcal{B}/I \), since \( \mathcal{C} \setminus N_\xi \) is \( \sigma \)-compact and \( \mathcal{B}/I \) is a \( \sigma \)-algebra. Let \( A_\xi \subset D_\xi \) be such that \( A_\xi = \{ B/I : B \in A_\xi \} \) is a maximal antichain in \( \mathcal{B}/I \). It is enough to show that no filter intersects all \( A_\xi \)'s. But if there were a filter \( F \) in \( \mathcal{B}/I \) intersecting all \( A_\xi \)'s then for every \( \xi < \omega_1 \) there would exist a \( B_\xi \in A_\xi \) with \( B_\xi/I \in F \cap A_\xi \). Thus, the set \( \bigcap_{\xi < \omega_1} \text{cl}(B_\xi) \) would be non-empty, despite the fact that it is disjoint from \( \bigcup_{\xi < \omega_1} N_\xi = \mathcal{C} \).

To finish the proof it is enough to show that \( (\ast) \) follows from \( \text{CPA}_{\text{cube}} \). But this follows immediately from the fact that any cube \( P \) in \( \mathcal{C} \) contains a subcube \( Q \in I \) as any cube \( P \) can be partitioned into \( \xi \)-many disjoint subcubes and, by the ccc property of \( I \), only countably many of them can be outside \( I \).

The authors like to thank professor Anastasis Kamburelis for many helpful comments concerning the results presented here. We are especially grateful him for pointing out a gap in an earlier version of the proof of Theorem 6.2.

Other consequences of \( \text{CPA}_{\text{cube}} \) can be found in [6], [11], [18], and in the monograph in preparation [7].

References


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2Preprints marked by * are available in electronic form from Set Theoretic Analysis Web Page: http://www.math.wvu.edu/~kcies/STA/STA.html
