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The talk is based on the following papers:


Abstract

Definition (#6)

A selfmap $f : X \to X$ on a metric space $\langle X, d \rangle$ is **Pointwise Contractive**, (PC) if for any $x \in X$ there exist an $\varepsilon_x > 0$ and a $\lambda_x \in [0, 1)$ such that

$$d(f(x), f(y)) \leq \lambda_x d(x, y) \text{ for every } y \in B(x, \varepsilon_x).$$

The class PC is also known as Local Radial Contractions.

We discuss the following:

Theorem (KC & JJ, 2016)

*If $X$ is a compact rectifiably path connected space and $f : X \to X$ is a pointwise contractive map then $f$ has a unique fixed point, that is, there exists a unique point $\xi \in X$ such that $f(\xi) = \xi$.*

We will present it in the context of classical fixed point results for other classes of local and pointwise contractive maps.
A function $f : X \to X$ is called **Contractive**, $(C)$, if there exists a constant $0 \leq \lambda < 1$ such that for any two elements $x, y \in X$ we have $d(f(x), f(y)) \leq \lambda d(x, y)$.

**Theorem (Banach, 1922)**

*If $(X, d)$ is a complete metric space and $f : X \to X$ is $(C)$, then $f$ has a unique fixed point.*
A function $f : X \rightarrow X$ is called **Shrinking, (S)**, if for any two elements $x, y \in X, x \neq y$ we have $d(f(x), f(y)) < d(x, y)$.

**Theorem (Edelstein, 1962)**

If $\langle X, d \rangle$ is compact and $f : X \rightarrow X$ is (S), then $f$ has a unique fixed point.
Definition (#3)

A function \( f : X \to X \) is called \textit{Locally Shrinking, (LS)}, if for any element \( z \in X \) there exists an \( \varepsilon_z > 0 \) such that \( f \mid B(z, \varepsilon_z) \) is shrinking, i.e. for any two \( x \neq y \in B(z, \varepsilon_z) \) we have \( d(f(x), f(y)) < d(x, y) \).

Theorem (Edelstein, 1962)

Let \( \langle X, d \rangle \) be \textit{compact} and let \( f : X \to X \).

(i) If \( f \) is (LS), then \( f \) has a periodic point.

(ii) If \( f \) is (LS) and \( X \) is \textit{connected}, then \( f \) has a unique fixed point.
A function \( f : X \rightarrow X \) is called \textit{uniformly Pointwise Contracting}, \textit{(uPC)}, (a.k.a. uniform Local Radial Contractions) if there exists a \( \lambda \in [0, 1) \) such that for every \( z \in X \) there exists an \( \varepsilon_z > 0 \) such that for any element \( x \in B(z, \varepsilon_z) \) we have \( d(f(x), f(z)) \leq \lambda d(x, z) \).

**Theorem (Hu and Kirk, 1978; proof corrected by Jungck, 1982)**

If \( \langle X, d \rangle \) is a \textit{rectifiably path connected complete metric space} and a map \( f : X \rightarrow X \) is \textit{(uPC)}, then \( f \) has a unique fixed point.
Definition (#5)

A function $f : X \to X$ is called **Uniformly Locally Contracting**, (ULC), if there exist a $\lambda \in [0, 1)$ and an $\varepsilon > 0$ such that for every $z \in X$ the restriction $f \mid B(z, \varepsilon)$ is contractive with that $\lambda$.

Theorem (Edelstein, 1961)

Assume that $\langle X, d \rangle$ is complete and that $f : X \to X$ is (ULC). If $X$ is connected, then $f$ has a unique fixed point.
Classes of *Locally*, (L) (two variables) OR *Pointwise*, (P) (one variable)
AND *Contractive*, (C) (with $\lambda$) OR *Shrinking*, (S) (no $\lambda$)
make the following diagram:

\[
\begin{array}{c}
(C) \quad (ULC) \quad (uLC) \quad (LC) \\
(S) \quad (ULS) \quad (LS) \quad (PC) \\
(UPC) \quad (uPC) \quad (PS) \\
(UPS) \end{array}
\]

**Remark:** (ULS) = (UPS) and (ULC) = (UPC).
Local Properties

Therefore,

\[(C) \to (ULC) \to (uLC) \to (LC)\]

\[(S) \to (ULS) \to (LS) \to (PS)\]

\[(uPC) \to (PC)\]

is the real picture.
So where are the fixed point theorems?
Of course it depends on the space \(X\).
All spaces $X$ are assumed to be complete.

Edelstein \((LS) + X\) compact and connected.
Edelstein (LS) + $X$ compact and connected
Edelstein (ULC) + $X$ connected
Summary of Local Properties with Fixed Points

- Edelstein: (LS) + $X$ compact and connected
- Edelstein: (ULC) + $X$ connected
- Hu & Kirk: (uPC) & $X$ rectifiably path connected (r.p.c.)
Recall,

**Definition (#6)**

A function \( f : X \rightarrow X \) is called *Pointwise Contractive, (PC)*, if for every \( z \in X \) there exist \( \lambda_z \in [0, 1) \) and an \( \varepsilon_z > 0 \) such that
\[
d(f(x), f(z)) \leq \lambda d(x, z)
\]
whenever \( x \in B(z, \varepsilon) \).

**Theorem (C & J, Top. and its App. 204 2016 70-78)**

Assume that \( \langle X, d \rangle \) is *compact and rectifiably path connected*. If \( f : X \rightarrow X \) is (PC), then \( f \) has a unique fixed point.
Edelstein (LS) and $X$ compact and connected
Edelstein (ULC) and $X$ connected
Hu & Kirk (uPC) and $X$ rectifiably path connected.
KC & JJ (PC) and $X$ rectifiably path connected and compact
Necessity of compactness of $X$

Recall,

**Theorem (KC & JJ, 2016)**

*If $X$ is a compact rectifiably path connected space and $f : X \to X$ is a (PC) map then $f$ has a unique fixed point, that is, there exists a unique point $\xi \in X$ such that $f(\xi) = \xi$.***

**Example (Munkres, 1974)**

The map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{2} \left( x + \sqrt{x^2 + 1} \right)$ has no periodic points because $f(x) > x$ for all $x \in X$. It is (PC), in fact, $f$ is from the class $(S) \cap (LC)$. This follows from the MVT because

$$f'(x) = \frac{1}{2} \left( 1 + \frac{x}{\sqrt{x^2+1}} \right)$$

hence, for any $a \in \mathbb{R}$,

$$f'[-\infty, a] = (0, c]$$

for some $c \in (0, 1)$. 
Necessity of compactness $X$

**Definition (#7)**

A map $f : X \to X$ is *uniformly Locally Contractive* (uLC) if there exists a $\lambda \in [0, 1)$ such for every $z \in X$ there is an $\varepsilon_x > 0$ so that $d(f(x), f(y)) \leq \lambda d(x, y)$ for any two $x, y \in B(z, \varepsilon_x)$.

**Example (Rakotch, 1962)**

There exists a closed, connected subset $X \subset \mathbb{R}^2$ and a map $f : X \to X$ which is (uLC) but every forward-orbit of $f$ is divergent. So $f$ has no periodic points.
Necessity of connectedness of $X$

For a selfmap $f$ on $\langle X, d \rangle$ and a limit point $x \in X$, let

$$D^* f(x) = \limsup_{y \to x} \frac{d(f(x), f(y))}{d(x, y)},$$

and for isolated point $x$ we set $D^* f(x) = 0$. $D^*$ is called as absolute derivative by Charatonik and Insall.

Remark

For $f : X \to X$,

- $f$ is (PC) iff $D^* f(x) < 1$ for all $x \in X$.
- $f$ is (uPC) iff $\sup \{D^* f(x) : x \in X\} < 1$

Example (KC & JJ, 2016)

There exists a compact perfect (Cantor-like) set $\mathcal{X} \subseteq \mathbb{R}$ and an auto-homeomorphism $f : \mathcal{X} \to \mathcal{X}$ with $D^* f' \equiv 0$ (so $f$ is (uPC) with any $\lambda \in (0, 1)$) and without periodic points.
No periodic points Examples

(C) \rightarrow (ULC) \rightarrow (uLC)_{conn.}^{Rakotch} \rightarrow (LC)

(S) \rightarrow (ULS) \rightarrow (LS)

(uPC)_{comp.}^{CJ} \rightarrow (PC)

(PS)
Problem (1)

Assume that $\langle X, d \rangle$ is compact and either connected or path connected. If the map $f : \langle X, d \rangle \to \langle X, d \rangle$ is (PC), must $f$ have either fix or periodic point? What if $f$ is (PS)?
Problem (2)

Assume that $\langle X, d \rangle$ is compact and either connected or path connected. If the map $f : \langle X, d \rangle \to \langle X, d \rangle$ is (uPC), must $f$ have either fix or periodic point?
Problem (3)

Assume that \( \langle X, d \rangle \) is compact and rectifiably path connected. If the map \( f : \langle X, d \rangle \rightarrow \langle X, d \rangle \) is (PS), must \( f \) have either fix or periodic point?
Recall, 

**Theorem (C & J, 2016)**

Assume that $\langle X, d \rangle$ is compact and rectifiably path connected. If $f : X \to X$ is (PC), then $f$ has a unique fixed point.

**PROOF (outline).** For $x, y \in X$ and a rectifiable path $p : [a, b] \to X$, $p(a) = x$, $p(b) = y$ let

$$\ell(p) = \sup\left\{ \sum_{i<n} d(t_i, t_{i+1}) : n < \omega \text{ and } a = t_0 < t_1 < \ldots < t_n = b \right\}.$$

Define $D_0 : X^2 \to [0, \infty)$,

$$D_0(x, y) = \inf\{ \ell(p) : p \text{ is a rectifiable path from } x \text{ to } y \}.$$

We need to show:

1. $D_0$ is a metric on $X$;
2. $\langle X, D_0 \rangle$ is complete;
3. There exists $\bar{x} \in X$ such that
   $$D_0(\bar{x}, f(\bar{x})) = L = \inf\{ D_0(x, f(x)) : x \in X \};$$
4. $L = 0$. 

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Locally contractive maps and fixed point theorems
Even when $\langle X, d \rangle$ is compact, $\langle X, D_0 \rangle$ does not need to be. Let $X$ be the Topologist’s Sine Curve with arc. Then $\langle X, d \rangle$ with standard metric from $\mathbb{R}^2$, is compact but $\langle X, D_0 \rangle$ is not. It’s actually homeomorphic with $[0, \infty)$. So (3) is not obvious.

Figure: Topologist’s Sine Curve with arc.
Proof of (3)

(3) There exists $\bar{x} \in X$ such that
$$D_0(\bar{x}, f(\bar{x})) = L = \inf \{ D_0(x, f(x)) : x \in X \}.$$ 

Let $\langle x_n \in X : n < \omega \rangle$ be a sequence with $\lim_{n \to \infty} D_0(x_n, f(x_n)) = L$. We have:

**Theorem (Menger 1930)**

*In a metric space $X$, if there is a rectifiable path in $X$ from $x$ to $y$, then there is a geodesic, i.e. a path with minimal length $\ell$, in $X$ from $x$ to $y$.*

so for every $n < \omega$ there exists a path $p_n : [0, 1] \to X$ from $x_n$ to $f(x_n)$ with range $P_n \subseteq X$ and $\ell(p_n) = D_0(x_n, f(x_n))$. 
We have the following:

**Theorem (Myers 1945)**

Let \( \langle X, d \rangle \) be a compact metric space and, for any \( n < \omega \), let \( p_n : [0, 1] \to X \) be a rectifiable path such that \( \ell(p_n \upharpoonright [0, t]) = t \ell(p_n) \) for any \( t \in [0, 1] \). If \( L = \lim \inf_{n \to \infty} \ell(p_n) < \infty \), then there exists a subsequence \( \langle p_{n_k} : k < \omega \rangle \) converging uniformly to a rectifiable path \( p : [0, 1] \to X \) with \( \ell(p) \leq L \).

WLOG, by reparametrizing our \( p_n \), we can assume that for any \( t \in [0, 1] \), \( \ell(p_n \upharpoonright [0, t]) = t \ell(p_n) \).

So by the Myers’ Theorem there exists a subsequence \( \langle p_{n_k} : k < \omega \rangle \) converging uniformly to a rectifiable path \( p : [0, 1] \to X \) with \( \ell(p) \leq L \).

Take \( \bar{x} = p(0) = \lim_{k \to \infty} p_{n_k}(0) = \lim_{k \to \infty} x_{n_k} \), then \( p \) is from \( \bar{x} \) to \( p(1) = \lim_{k \to \infty} p_{n_k}(1) = \lim_{k \to \infty} f(x_{n_k}) = f(\bar{x}) \).

So, \( D_0(\bar{x}, f(\bar{x})) \leq \ell(p) \leq L \), that is, \( \bar{x} \) satisfies (3).
Thank you for your attention.