Extending connectivity functions on $\mathbb{R}^n$

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Abstract

A function $f : \mathbb{R}^n \to \mathbb{R}$ is a connectivity function if for every connected subset $C$ of $\mathbb{R}^n$ the graph of the restriction $f \mid C$ is a connected subset of $\mathbb{R}^{n+1}$, and $f$ is an extendable connectivity function if $f$ can be extended to a connectivity function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\mathbb{R}^n$ embedded into $\mathbb{R}^{n+1}$ as $\mathbb{R}^n \times \{0\}$. There exists a connectivity function $f : \mathbb{R} \to \mathbb{R}$ that is not extendable. We prove that for $n \geq 2$ every connectivity function $f : \mathbb{R}^n \to \mathbb{R}$ is extendable. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Given functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^{n+1} \to \mathbb{R}$, we say that $g$ extends $f$ if $g$ extends the composition $f \circ \tau : \mathbb{R}^n \times \{0\} \to \mathbb{R}$, where $\tau : \mathbb{R}^n \times \{0\} \to \mathbb{R}^n$ and

$$\tau((x_1, x_2, \ldots, x_n, 0)) = (x_1, x_2, \ldots, x_n),$$

for every $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is a connectivity function if for every connected subset $C$ of $\mathbb{R}^n$ the graph of the restriction $f \mid C$ is a connected subset of $\mathbb{R}^{n+1}$, and $f$ is an extendable connectivity function if there exists a connectivity function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ extending $f$. 
It follows immediately from the definition that every extendable connectivity function is a connectivity function. Cornette [3] and Roberts [9] proved that there exists a connectivity function \( f : \mathbb{R} \rightarrow \mathbb{R} \) that is not extendable. This result was surprising and sparked the interest in the family of extendable connectivity functions. Ciesielski and Wojciechowski [2] asked whether there exists a connectivity function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), with \( n \geq 2 \), that is not extendable. In this paper we will show that the answer to that question is negative.

**Theorem 1.** If \( n \geq 2 \) then every connectivity function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is extendable.

To prove Theorem 1 we will use ideas from Gibson and Roush [5] where is formulated a necessary and sufficient condition for a connectivity function \( f : [0, 1] \rightarrow [0, 1] \) to be extendable to a connectivity function \( f : [0, 1]^2 \rightarrow [0, 1] \) (if one considers \([0, 1] \) to be embedded in \([0, 1]^2 \) as \([0, 1] \times [0, 1] \)).

Our basic terminology and notation is standard. (See [1] or [4].) In particular, if \( A \) is a subset of a metric space \( X \), then \( \text{bd} A, \text{cl} A \) and \( \text{diam} A \) will denote the boundary, closure, and diameter of \( A \) in \( X \), respectively, and if \( f \) is a function and \( A \) is a subset of its domain, then \( f[A] \) is the image of \( A \) under \( f \).

The following additional terminology will be useful in our proof. Given a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), a peripheral pair (for \( f \)) is an ordered pair \( \langle A, I \rangle \) with \( I \) being a closed interval in \( \mathbb{R} \) and \( A \) being an open bounded subset of \( \mathbb{R}^n \) with \( f[\text{bd} A] \subseteq I \). Given \( \varepsilon > 0 \), an \( \varepsilon \)-peripheral pair is a peripheral pair \( \langle A, I \rangle \) with \( \text{diam} A < \varepsilon \) and \( \text{diam} I < \varepsilon \). Given a point \( x \in \mathbb{R}^n \), a peripheral pair for \( f \) at \( x \) is a peripheral pair \( \langle A, I \rangle \) for \( f \) with \( x \in A \) and \( f(x) \in I \). A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is said to be peripherally continuous if for every \( x \in \mathbb{R}^n \) and \( \varepsilon > 0 \) there is an \( \varepsilon \)-peripheral pair for \( f \) at \( x \).

The class of peripherally continuous functions \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is strictly larger than the class of connectivity functions. However, the following result holds.

**Theorem 2.** If \( n \geq 2 \) then a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is peripherally continuous if and only if it is a connectivity function.

The implication that a connectivity function is peripherally continuous in Theorem 2 was proved by Hamilton [7] and Stallings [10], and the opposite implication was proved by Hagan [6].

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a function and \( \mathcal{P} \) be a family of peripheral pairs for \( f \). We say that \( \mathcal{P} \) locally converges to 0 if for every \( \varepsilon > 0 \) and every bounded set \( X \subseteq \mathbb{R}^n \) the set

\[
\{ \langle A, I \rangle \in \mathcal{P} : A \cap X \neq \emptyset \text{ and } \text{diam} A \geq \varepsilon \}
\]

is finite, and that \( \mathcal{P} \) has the intersection property provided \( I \cap I' \neq \emptyset \) for any \( \langle A, I \rangle, \langle A', I' \rangle \in \mathcal{P} \) such that each of the sets \( A \cap A' \), \( A \setminus A' \), and \( A' \setminus A \) is nonempty. Given \( X \subseteq \mathbb{R}^n \), we say that \( \mathcal{P} \) is an \( f \)-base for \( X \) if for every \( \varepsilon > 0 \) and \( x \in X \) there exists an \( \varepsilon \)-peripheral pair for \( f \) at \( x \) that belongs to \( \mathcal{P} \). Note that a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is peripherally continuous if and only if there exists an \( f \)-base for some set \( X \subseteq \mathbb{R}^n \) that
contains all points of discontinuity of $f$. A peripheral family for $f : \mathbb{R}^n \to \mathbb{R}$ is a countable family of peripheral pairs for $f$ that locally converges to 0, has the intersection property, and is an $f$-base for $\mathbb{R}^n$.

Theorem 1 follows from Theorem 2 and the following two results.

**Theorem 3.** If $n \geq 2$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a peripherally continuous function, then there exists a peripheral family for $f$.

If $(A, I)$ is a peripheral pair (for some $f : \mathbb{R}^n \to \mathbb{R}$), then the cylindrical extension of $(A, I)$ is a pair $(A', I)$, where

$$A' = A \times (-\text{diam} A, \text{diam} A) \subseteq \mathbb{R}^{n+1}.$$  

If $\mathcal{P}$ is a set of peripheral pairs, then the cylindrical extension of $\mathcal{P}$ is the set of cylindrical extensions of all the elements of $\mathcal{P}$.

The case $n = 1$ of the following theorem is a modification of a result of Gibson and Roush [5].

**Theorem 4.** If $n \geq 1$ and $\mathcal{P}$ is a peripheral family for $f : \mathbb{R}^n \to \mathbb{R}$, then there exists a continuous function $h : \mathbb{R}^{n+1} \setminus (\mathbb{R}^n \times \{0\}) \to \mathbb{R}$ such that every element of the cylindrical extension of $\mathcal{P}$ is a peripheral pair for the function $g = h \cup (f \circ \tau) : \mathbb{R}^{n+1} \to \mathbb{R}$, where $\tau : \mathbb{R}^n \times \{0\} \to \mathbb{R}^n$ is the bijection as in (1).

The proof of Theorem 3 is given in Section 2, and the proof of Theorem 4 can be found in Section 3. Now we shall give the proof of Theorem 1.

**Proof of Theorem 1.** Let $n \geq 2$ and $f : \mathbb{R}^n \to \mathbb{R}$ be a connectivity function. Since $f$ is peripherally continuous, it follows from Theorem 3 that there exists a peripheral family $\mathcal{P}$ for $f$. Let $\mathcal{Q}$ be the cylindrical extension of $\mathcal{P}$. By Theorem 4 there exists a function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $g$ extends $f$, the restriction of $g$ to $\mathbb{R}^{n+1} \setminus (\mathbb{R}^n \times \{0\})$ is continuous, and every element of $\mathcal{Q}$ is a peripheral pair for $g$. The proof will be complete when we show that $\mathcal{Q}$ is a $g$-base for $\mathbb{R}^n \times \{0\}$ since then it will follow that $g$ is peripherally continuous and hence a connectivity function.

Let $\varepsilon > 0$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Since $\mathcal{P}$ is an $f$-base for $\mathbb{R}^n$, there is $(A, I) \in \mathcal{P}$ such that $\text{diam} A < \varepsilon/\sqrt{3}$, $\text{diam} I < \varepsilon$, $x \in A$, and $f(x) \in I$. Then the cylindrical extension $(A', I) \in \mathcal{Q}$ of $(A, I)$ is an $\varepsilon$-peripheral pair for $g$ at $\bar{x} = (x_1, \ldots, x_n, 0)$ implying that $\mathcal{Q}$ is a $g$-base for $\mathbb{R}^n \times \{0\}$.  \qed
2. Peripheral families for connectivity functions

In this section we are going to prove Theorem 3. First, let us introduce some more terminology. Throughout this section we will assume that \( n \) is a fixed integer and that \( n \geq 2 \).

Given \( X, Y \subseteq \mathbb{R}^n \), the boundary of \( X \cap Y \) in \( X \) will be denoted by \( \text{bd}_X Y \). The inductive dimension \( \text{ind} X \) of a subset \( X \subseteq \mathbb{R}^n \) is defined inductively as follows. (See, for example, Engelking [4].)

(i) \( \text{ind} X = -1 \) if and only if \( X = \emptyset \).
(ii) \( \text{ind} X \leq m \) if for any \( p \in X \) and any open neighborhood \( W \) of \( p \) there exists an open neighborhood \( U \subseteq W \) of \( p \) such that \( \text{ind} \text{bd}_X U \leq m - 1 \).
(iii) \( \text{ind} X = m \) if \( \text{ind} X \leq m \) and it is not true that \( \text{ind} X \leq m - 1 \).

A fundamental result of dimension theory states that \( \text{ind} \mathbb{R}^n = n \).

Given a set \( A \subseteq \mathbb{R}^n \) and an integer \( m \geq 1 \), we say that \( A \) is an \( m \)-dimensional Cantor manifold if \( A \) is compact, \( \text{ind} A = m \), and for every \( X \subseteq A \) with \( \text{ind} X \leq m - 2 \), the set \( A \setminus X \) is connected. (See [8].) Given a subset \( A \) of \( \mathbb{R}^n \), we say that \( A \) is a quasiball if \( A \) is a bounded and connected open set, and \( \text{bd} A \) is an \((n - 1)\)-dimensional Cantor manifold. (See [2].) A peripheral pair \( \langle A, I \rangle \) with \( A \) being a quasiball will be called a nice peripheral pair. Given \( \varepsilon, \delta > 0 \), an \( \langle \varepsilon, \delta \rangle \)-peripheral pair is a peripheral pair \( \langle A, I \rangle \) with \( \text{diam} A < \varepsilon \) and \( \text{diam} I < \delta \). The following theorem follows immediately from Corollary 5.5 in [2].

**Theorem 5.** If \( f : \mathbb{R}^n \to \mathbb{R} \) is a peripherally continuous function, then for any \( \varepsilon, \delta > 0 \) and \( x \in \mathbb{R}^n \) there exists a nice \( \langle \varepsilon, \delta \rangle \)-peripheral pair for \( f \) at \( x \).

We say that quasiballs \( A \) and \( A' \) are independent if each of the sets \( A \cap A' \), \( A \setminus A' \), and \( A' \setminus A \) is nonempty. The following lemma is a restatement of Lemma 5.6 in [2].

**Lemma 6.** If \( A \) and \( A' \) are independent quasiballs in \( \mathbb{R}^n \), then \( \text{bd} A \cap \text{bd} A' \neq \emptyset \).

The following lemma follows immediately from Lemma 6.

**Lemma 7.** If \( \mathcal{P} \) is a family of nice peripheral pairs, then \( \mathcal{P} \) has the intersection property.

For every positive integer \( i \in \mathbb{N} \), let

\[
D_i = \left\{ \frac{-4i^2}{4i}, \frac{-4i^2 + 1}{4i}, \ldots, \frac{4i^2}{4i} \right\}
\]

and

\[
J_i = \{ J_{i,q} : q \in D_i \},
\]

where \( J_{i,q} \) is the open interval

\[
J_{i,q} = \left( q - \frac{1}{4i}, q + \frac{1}{4i} \right),
\]

for each \( q \in D_i \).
Lemma 8. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a function and, for every $i \in \mathbb{N}$ and $q \in D_i$, let

$$\mathcal{P}_{i,q} = \{ (A_y, I_y) : y \in \Gamma_{i,q} \}$$

be a family of $(1/i)$-peripheral pairs for $f$ such that

$$f^{-1}(J_{i,q}) \subseteq \bigcup_{y \in \Gamma_{i,q}} A_y \quad \text{and} \quad J_{i,q} \subseteq \bigcap_{y \in \Gamma_{i,q}} I_y.$$ Then

$$\mathcal{P} = \bigcup_{i \in \mathbb{N}} \bigcup_{q \in D_i} \mathcal{P}_{i,q}$$

is an $f$-base for $\mathbb{R}^n$.

Proof. Let $\varepsilon > 0$ and $x \in \mathbb{R}^n$. Then there are $i \in \mathbb{N}$ and $q \in D_i$ with $1/i \leq \varepsilon$ and $f(x) \in J_{i,q}$. Since

$$f^{-1}(J_{i,q}) \subseteq \bigcup_{y \in \Gamma_{i,q}} A_y,$$

there is $\delta \in \Gamma_{i,q}$ such that $x \in A_\delta$. Since

$$J_{i,q} \subseteq \bigcap_{y \in \Gamma_{i,q}} I_y,$$

it follows that $(A_\delta, I_\delta)$ is an $\varepsilon$-peripheral pair for $f$ at $x$.

Now we are ready to prove Theorem 3.

Proof of Theorem 3. Let $n \geq 2$ and $f: \mathbb{R}^n \to \mathbb{R}$ be a peripherally continuous function. Fix $i \in \mathbb{N}$ and $q \in D_i$. By Theorem 5 for each $x \in f^{-1}(J_{i,q})$ there exists a nice $(1/i, 1/4i)$-peripheral pair $(A_{i,q,x}, I_{i,q,x})$ for $f$ at $x$. Let

$$\mathcal{T}_{i,q} = \{ (A_{i,q,x}, \text{cl}(I_{i,q,x} \cup J_{i,q})) : x \in f^{-1}(J_{i,q}) \}.$$ Note that since

$$f(x) \in I_{i,q,x} \cap J_{i,q} \neq \emptyset$$

for every $x \in f^{-1}(J_{i,q})$, the elements of $\mathcal{T}_{i,q}$ are $(1/i, 3/4i)$-peripheral pairs for $f$.

Let $j, k \in \mathbb{N}$ be any positive integers with $j > i$. Set

$$\mathcal{T}_{i,q}^k = \{ (A, I) \in \mathcal{T}_{i,q} : A \cap B_k \neq \emptyset \text{ and } A \cap B_{k'} = \emptyset \text{ for every } k' < k \},$$

where $B_k$ is the open ball of center $(0, 0, \ldots, 0)$ and radius $k$, and

$$\mathcal{T}_{i,q}^{k,j} = \{ (A, I) \in \mathcal{T}_{i,q}^k : \frac{1}{j} \leq \text{diam } A < \frac{1}{j-1} \}.$$ Moreover, let

$$\mathcal{C}_{i,q} = \text{cl} \left( \bigcup_{(A, I) \in \mathcal{T}_{i,q}^{k,j}} A \right).$$
and
\[ E_{i,q}^{k,j} = C_{i,q}^{k,j} \setminus \bigcup_{(A,I) \in T_{i,q}^{k,j}} A. \]

Fix \( y \in E_{i,q}^{k,j} \). Let \( \langle A_y, I'_y \rangle \) be a nice \((1/j, 1/4i)\)-peripheral pair for \( f \) at \( y \). Since
\[ E_{i,q}^{k,j} \subseteq \text{cl} \left( \bigcup_{(A,I) \in T_{i,q}^{k,j}} \text{bd} A \right), \]
there is \( (A, I) \in T_{i,q}^{k,j} \) such that
\[ A_y \cap \text{bd} A \neq \emptyset. \]

Since \( \text{diam} A_y < \text{diam} A \), it follows that the quasiballs \( A \) and \( A_y \) are independent and so Lemma 6 implies that \( I \cap I'_y \neq \emptyset \). Let \( I_y = I \cup I'_y \) for every \( y \in E_{i,q}^{k,j} \) and
\[ S_{i,q}^{k,j} = T_{i,q}^{k,j} \cup \{\langle A_y, I_y \rangle; \ y \in E_{i,q}^{k,j}\}. \]

Note that \( J_{i,q} \subseteq I \) for every \( (A,I) \in S_{i,q}^{k,j} \). Since the set \( C_{i,q}^{k,j} \) is compact and
\[ C_{i,q}^{k,j} \subseteq \bigcup_{(A,I) \in S_{i,q}^{k,j}} A, \]
there is a finite subset \( P_{i,q}^{k,j} \) of \( S_{i,q}^{k,j} \) such that
\[ C_{i,q}^{k,j} \subseteq \bigcup_{(A,I) \in P_{i,q}^{k,j}} A. \]

Let
\[ P_{i,q} = \bigcup_{k \in \mathbb{N}} \bigcup_{j > i} P_{i,q}^{k,j} = \{\langle A_y, I_y \rangle; \ y \in E_{i,q}^{k,j}\}. \]

It is clear that the elements of \( P_{i,q} \) are \((1/i)\)-peripheral pairs and
\[ J_{i,q} \subseteq \bigcap_{y \in E_{i,q}^{k,j}} I_y. \]

Moreover,
\[ f^{-1}(J_{i,q}) \subseteq \bigcup_{(A,I) \in \Gamma_{i,q}} A \subseteq \bigcup_{(A,I) \in \Gamma_{i,q}} \text{cl} A \subseteq \bigcup_{k \in \mathbb{N}} \bigcup_{j > i} C_{i,q}^{k,j} \subseteq \bigcup_{y \in E_{i,q}^{k,j}} A_y, \]

implying, by Lemma 8, that
\[ P = \bigcup_{i \in \mathbb{N}} \bigcup_{q \in D_i} P_{i,q} \]
is an \( f \)-base for \( \mathbb{R}^n \). Of course \( P \) is countable and since all peripheral pairs in \( P \) are nice, it follows from Lemma 7 that \( P \) has the intersection property. It remains to prove the following claim.

**Claim.** The family \( P \) locally converges to 0.
We are going now to prove the claim. First note that if \((A, I) \in T_{i,q}^{k,j}\) and \(k' < k\), then \(A \cap B_{k'} = \emptyset\), implying that \(y \notin B_{k'}\) (and hence \(A_y \not\subseteq B_{k'}\)) for any \(y \in E_{i,q}^{k,j}\). Therefore
\[
A \not\subseteq B_{k'} \quad \text{for any} \quad (A, I) \in S_{i,q}^{k,j} \quad \text{and} \quad k' < k.
\]
Also note that
\[
diam A < \frac{1}{j'} \quad \text{for any} \quad (A, I) \in S_{i,q}^{k,j} \quad \text{and} \quad j' < j. \tag{3}
\]
Now let \(\varepsilon > 0\) and \(X \subseteq \mathbb{R}^n\) be a bounded set. Then there are \(j', k' \in \mathbb{N}\) such that \(1/j' < \varepsilon\) and \(X\) is a subset of the ball \(B_{k'-1}\). Let \((A, I) \in \mathcal{P}\) be such that \(A \cap X \neq \emptyset\) and \(diam A \geq \varepsilon\). Since \(A \cap B_{k'-1} \neq \emptyset\) and \(diam A < 1\), it follows that \(A \subseteq B_{k'}\). Therefore, since \(diam A \geq 1/j'\), it follows from (2) and (3) that if \((A, I) \in \mathcal{P}_{i,q}^{k,j} \subseteq S_{i,q}^{k,j}\), then \(k \leq k'\) and \(j \leq j'\). Thus
\[
(A, I) \in \mathcal{P}^{k,j'} \subseteq \bigcup_{k' \leq k} \bigcup_{j' \leq j} \bigcup_{i,q} \mathcal{P}_{i,q}^{k,j}. 
\]
Since the set \(\mathcal{P}^{k,j'}\) is finite, the proof of the claim, and hence of the theorem is complete. \(\square\)

3. Connectivity functions are extendable

In this section we are going to prove Theorem 4.

A partial order on a set \(T\) is a binary relation \(\preceq\) on \(T\) that is reflexive, transitive and antisymmetric (that is, \(t \preceq s\) and \(s \preceq t\) imply \(t = s\) for every \(s, t \in T\)). We say that \(\preceq\) has the finite predecessor property if for every \(t \in T\) the set \(\{s \in T: s \preceq t\}\) of \(\preceq\)-predecessors of \(t\) is finite. A partial order \(\preceq^\ast\) on a set \(T\) is an \(\omega\)-order if there is a bijection \(f: \omega \rightarrow T\) (where \(\omega = \{0, 1, \ldots\}\)) such that \(f(t) \preceq^\ast f(s)\) if and only if \(t \preceq s\). Given partial orders \(\preceq\) and \(\preceq^\ast\) on \(T\), we say that \(\preceq^\ast\) extends \(\preceq\) if and only if \(t \preceq s\) implies \(t \preceq^\ast s\) for every \(s, t \in T\).

**Lemma 9.** If \(\preceq\) is a partial order on an infinite countable set \(T\) with the finite predecessor property, then there is an \(\omega\)-order \(\preceq^\ast\) on \(T\) that extends \(\preceq\).

**Proof.** It is enough to show that there is a bijection \(f: \omega \rightarrow T\) such that \(f(i) \preceq f(j)\) implies \(i \preceq j\). Let \(\preceq\) be any fixed \(\omega\)-order on \(T\). We shall define the value \(f(i)\) by induction on \(i\). Let \(i \in \omega\) and assume that \(f(j)\) has been defined for every \(j < i\). Let
\[
T_i = T \setminus \{f(j): j < i\},
\]
and let \(T'_i\) consist of all \(\preceq\)-minimal elements in \(T_i\). For every \(t \in T_i\) the set of \(\preceq\)-predecessors of \(t\) is finite so there is \(s \in T'_i\) with \(s \preceq t\). In particular, \(T'_i\) is nonempty. Let \(f(i)\) be the \(\preceq\)-minimal element of \(T'_i\).

It is obvious from the construction that \(f\) is injective and that \(f(i) \preceq f(j)\) implies \(i \preceq j\) for every \(i, j \in \omega\). To see that \(f\) is surjective note that for any \(i \in \omega\) and \(t \in T_i\) the set of
≼-predecessors of \( t \) is finite, so one of them is in \( T'_i \). This predecessor of \( t \) will eventually become a value of \( f \) since \( \preceq \) is an \( \omega \)-order. Then the number of unassigned \( \preceq \)-predecessors of \( t \) becomes smaller and hence eventually \( t \) itself must become a value of \( f \). \( \Box \)

A family \( \mathcal{A} \) of subsets of a metric space \( X \) is \textit{locally finite} if for every \( x \in X \) some open neighborhood of \( x \) intersects only finitely many elements of \( \mathcal{A} \). Let a \textit{Tietze family} for a metric space \( X \) be a countable family

\[
\mathcal{F} = \{ (C_\gamma, I_\gamma) : \gamma \in \Gamma \}
\]

such that:

1. \( \mathcal{A} = \{ C_\gamma : \gamma \in \Gamma \} \) is a locally finite closed cover of \( X \) with any \( C_\gamma \) intersecting only finitely many elements of \( \mathcal{A} \);
2. for every \( \gamma \in \Gamma \), \( I_\gamma \) is either equal to \( \mathbb{R} \) or is a closed interval in \( \mathbb{R} \);
3. for every \( \Phi \subseteq \Gamma \)
   \[
   \text{if } \bigcap_{\gamma \in \Phi} C_\gamma \neq \emptyset \text{ then } \bigcap_{\gamma \in \Phi} I_\gamma \neq \emptyset.
   \]

The following result will be the key step in our proof of Theorem 4.

\textbf{Theorem 10.} Let \( X \) be a metric space and \( \mathcal{F} = \{ (C_\gamma, I_\gamma) : \gamma \in \Gamma \} \) be a Tietze family for \( X \). Then there is a continuous function \( h : X \to \mathbb{R} \) such that \( h[C_\gamma] \subseteq I_\gamma \) for every \( \gamma \in \Gamma \).

\textbf{Proof.} Let \( \mathcal{A} = \{ C_\gamma : \gamma \in \Gamma \} \), and

\[
T_\mathcal{A} = \left\{ \Phi \subseteq \Gamma : \bigcap_{\gamma \in \Phi} C_\gamma \neq \emptyset \right\}.
\]

Let \( \preceq_\mathcal{A} \) be the partial order of reversed inclusion on \( T_\mathcal{A} \), that is, \( \Phi_1 \preceq_\mathcal{A} \Phi_2 \) if and only if \( \Phi_2 \subseteq \Phi_1 \). Since every element of \( \mathcal{A} \) intersects only finitely many elements of \( \mathcal{A} \), it follows that the elements of \( T_\mathcal{A} \) are finite sets and that \( \preceq_\mathcal{A} \) has the finite predecessor property.

Let \( \preceq_\mathcal{A}^* \) be an \( \omega \)-order extending \( \preceq_\mathcal{A} \) and for every \( \Phi \in T_\mathcal{A} \) let

\[
C_\Phi = \bigcap_{\gamma \in \Phi} C_\gamma \neq \emptyset.
\]

Take the enumeration \( \Phi_1, \Phi_2, \ldots \) of \( T_\mathcal{A} \) with

\[
\Phi_1 \preceq_\mathcal{A}^* \Phi_2 \preceq_\mathcal{A}^* \cdots
\]

and for every \( i = 1, 2, \ldots \) let

\[
C_i = \bigcup_{j \leq i} C_{\Phi_j}, \quad C'_i = C_i \cap C_{\Phi_{i+1}},
\]

and

\[
I_i = \bigcap_{\gamma \in \Phi_i} I_\gamma \neq \emptyset.
\]
We are going to define a sequence \( h_1, h_2, \ldots \) of continuous functions \( h_i : C_i \rightarrow \mathbb{R} \) such that for every \( i = 1, 2, \ldots \) the function \( h_i+1 \) is an extension of \( h_i \) and

\[
h_i[C_i \cap C_{i+1}] \subseteq I_i
\]

for every \( i \in \Gamma \). Having defined such a sequence of functions our proof will be complete since it is easy to see that the function

\[
h = \bigcup_{i=1}^{\infty} h_i
\]

satisfies the required conditions. Indeed, (4) implies that \( h[C_{i+1}] \subseteq I_{i+1} \) for every \( i \in \Gamma \), and since \( \mathcal{F} \) is a locally finite closed cover of \( X \) it follows that \( h \) is a continuous function on \( X \).

Let \( h_1 : C_1 \rightarrow I_1 \) be any continuous function. Suppose that \( h_1 \) has been defined in such a way that (4) is satisfied. Let \( h_i \) be the restriction of \( h_1 \) to \( C_i \). It follows from (4) that \( h_i : C_i \rightarrow I_i \). Since \( C_i \) is a closed subset of \( C \), it follows from Tietze Extension Theorem that \( h_i \) can be extended to a continuous function \( h_i' : C_i \rightarrow I_i \). Let \( h_{i+1} = h_i \cup h_i' \). Since \( C_i \) and \( C_{i+1} \) are closed subsets of \( C \), the function \( h_{i+1} : C_i \rightarrow I_{i+1} \) is continuous. It remains to show that (4) is satisfied for \( h_{i+1} \).

Suppose that \( x \in C_i \cap C_{i+1} \). If \( x \in C_i \), then \( h_{i+1}(x) = h_i(x) \in I_i \) by the inductive hypothesis. Otherwise \( x \in C \) and so \( h_{i+1}(x) \in I_{i+1} \). Thus \( x \in C_i \cap C_{i+1} \), it follows that \( y \in C_i \) and so the proof is complete. \( \square \)

**Lemma 11.** Let \( n \geq 1 \), \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), \( \mathcal{P} \) be a peripheral family for \( f \), and \( Q \) be the cylindrical extension of \( \mathcal{P} \). If \( \{ (A_i, I_i) : 1 \leq i \leq k \} \subseteq Q \) and \( \text{bd} A_i \cap \text{bd} A_j \neq \emptyset \) for every \( i, j \leq k \), then \( \bigcap_{j=1}^{k} I_j \neq \emptyset \).

**Proof.** First we shall prove the lemma for \( k = 2 \). Suppose, by way of contradiction, that there exist \( (A_1, I_1), (A_2, I_2) \in Q \) with \( \text{bd} A_1 \cap \text{bd} A_2 \neq \emptyset \) and \( I_1 \cap I_2 = \emptyset \). Let \( (A'_1, I_1), (A'_2, I_2) \in \mathcal{P} \) be such that

\[
A_1 = A'_1 \times (-a_1, a_1) \quad \text{and} \quad A_2 = A'_2 \times (-a_2, a_2),
\]

where \( a_1 = \text{diam} A'_1 \) and \( a_2 = \text{diam} A'_2 \).

Since \( f[\text{bd} A'_1] \subseteq I_1 \) and \( f[\text{bd} A'_2] \subseteq I_2 \), we have

\[
\text{bd} A'_1 \cap \text{bd} A'_2 = \emptyset.
\]

It follows that \( A'_1 \cap A'_2 \neq \emptyset \) since otherwise we would have \( \text{cl} A'_1 \cap \text{cl} A'_2 = \emptyset \) in contradiction with \( \text{bd} A_1 \cap \text{bd} A_2 \neq \emptyset \). Since \( \mathcal{P} \) has the intersection property, one of \( A'_1, A'_2 \) is a subset of the other.
Assume that $A'_1 \subseteq A'_2$. Since $\text{cl} A'_1 \subseteq \text{cl} A'_2$, and $\text{bd} A'_1 \cap \text{bd} A'_2 = \emptyset$, it follows that $\text{cl} A'_1 \subseteq A'_2$. Since the set $\text{cl} A'_1$ is compact, there are $x_1, x_2 \in \text{cl} A'_1$ with $\text{diam} A'_1$ equal to the distance from $x_1$ to $x_2$. Since $x_1, x_2 \in A'_2$ and $A'_2$ is open, it follows that

$$a_1 = \text{diam} A'_1 < \text{diam} A'_2 = a_2.$$ 

and so

$$\text{bd} A_1 = \text{bd} A'_1 \times [-a_1, a_1] \cup A'_1 \times [-a_1, a_1] \subseteq A'_2 \times (-a_2, a_2) = A_2,$$

contradicting our assumption that $\text{bd} A_1 \cap \text{bd} A_2 \neq \emptyset$.

Now for $k > 2$ the assertion follows easily from the fact that if $\{I_j : 1 \leq j \leq k\}$ is a family of intervals in $\mathbb{R}$ and $I_j \cap I_m \neq \emptyset$ for every $j, m \leq k$, then $\bigcap_{j=1}^k I_j \neq \emptyset$. \(\square\)

Now we are ready to prove Theorem 4.

**Proof of Theorem 4.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, $\mathcal{P}$ be a peripheral family for $f$, $\mathcal{Q}$ be the cylindrical extension of $\mathcal{P}$, and

$$X = \mathbb{R}^{n+1} \setminus (\mathbb{R}^n \times \{0\}).$$

We need to construct a continuous function $h : X \to \mathbb{R}$ such that $h[\text{bd} A] \subseteq I$ for every $(A, I) \in \mathcal{Q}$. The existence of the function $h$ will follow from Theorem 10 after we have constructed a Tietze family

$$\mathcal{F} = \{(C_y, I_y) : y \in \Gamma'\}$$

for $X$ such that for every $(A, I) \in \mathcal{Q}$ there is $\Phi \subseteq \Gamma$ with

$$X \cap \text{bd} A \subseteq \bigcup_{\gamma \in \Phi} C_y \quad \text{and} \quad I_y = I \quad \text{for every } \gamma \in \Phi. \quad (5)$$

Let $\mathcal{K}$ consist of all closed intervals of the following forms: $[i, i + 1], [-i - 1, -i], [1/(i + 1), 1/i], \text{ and } [-1/i, -1/(i + 1)]$ for every $i = 1, 2, \ldots$. Set

$$\mathcal{A}_1 = \{(\text{cl} B^n_k \setminus B^n_{k-1}) \times [a, b] \subseteq \mathbb{R}^{n+1} : [a, b] \in \mathcal{K} \text{ and } k = 1, 2, \ldots\},$$

where $B^n_k \subseteq \mathbb{R}^n$ is the open ball with center $(0, 0, \ldots, 0)$ and radius $k$. Note that $\mathcal{A}_1$ is a locally finite closed cover of $X$.

Define

$$\mathcal{F}_1 = \{(C, \mathbb{R}) : C \in \mathcal{A}_1\}$$

and

$$\mathcal{F}_2 = \{(\text{bd} A \cap L, I) : (A, I) \in \mathcal{Q} \text{ and } L \in \mathcal{L}\},$$

where

$$\mathcal{L} = \{\mathbb{R}^n \times [a, b] : [a, b] \in \mathcal{K}\}.$$ 

Let $\Gamma_1$ and $\Gamma_2$ be disjoint sets of indices such that

$$\mathcal{F}_1 = \{(C_y, I_y) : y \in \Gamma_1\} \quad \text{and} \quad \mathcal{F}_2 = \{(C_y, I_y) : y \in \Gamma_2\}. $$
Obviously, for every $\langle A, I \rangle \in Q$ there is $\Phi \subseteq \Gamma_2$ such that (5) holds. Thus to complete the proof it remains to prove the following claim.

**Claim.** The family $F_1 \cup F_2$ is a Tietze family for $X$.

Let

$$A_2 = \{C_\gamma : \gamma \in \Gamma_2\}.$$  

Obviously, $A_1 \cup A_2$ is a closed cover of $X$. Since the family $\mathcal{P}$ is locally convergent to 0, every bounded subset of an element of $\mathcal{L}$ intersects only finitely many elements of $A_2$. Since each point $x \in X$ has an open neighborhood contained in at most two elements of $\mathcal{L}$, it follows that $A_2$ is locally finite, and hence $A_1 \cup A_2$ is locally finite.

Since every element $C$ of $A_1 \cup A_2$ is a bounded subset of an element of $\mathcal{L}$, it follows that $C$ intersects only finitely many elements in $A_2$, and it is clear that $C$ intersects only finitely many elements of $A_1$. Thus every element of $A_1 \cup A_2$ intersects only finitely many elements in $A_1 \cup A_2$.

Now suppose that

$$\bigcap_{\gamma \in \Phi_1 \cup \Phi_2} C_\gamma \neq \emptyset$$

for some $\Phi_1 \subseteq \Gamma_1$ and $\Phi_2 \subseteq \Gamma_2$. Since $\bigcap_{\gamma \in \Phi_2} C_\gamma \neq \emptyset$, it follows from Lemma 11 that $\bigcap_{\gamma \in \Phi_2} I_\gamma \neq \emptyset$. Since $I_\gamma = \mathbb{R}$ for $\gamma \in \Phi_2$, we have

$$\bigcap_{\gamma \in \Phi_1 \cup \Phi_2} I_\gamma = \bigcap_{\gamma \in \Phi_2} I_\gamma \neq \emptyset.$$  

Thus $F_1 \cup F_2$ is a Tietze family for $X$, and so the proof of the claim and hence of the theorem is complete. $\square$

**References**


