LONG SNAKES IN POWERS OF THE COMPLETE GRAPH
WITH AN ODD NUMBER OF VERTICES

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Abstract. In [5] Abbott and Katchalski ask if there exists a constant $c > 0$ such that for every $d \geq 2$ there is a snake (cycle without chords) of length at least $c3^d$ in the product of $d$ copies of the complete graph $K_3$. We show that the answer to the above question is positive, and that in general for any odd integer $n$ there is a constant $c_n$ such that for every $d \geq 2$ there is a snake of length at least $c_n n^d$ in the product of $d$ copies of the complete graph $K_n$.

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1. Introduction

By a path in a graph $G$ we mean a sequence of distinct vertices of $G$ with every pair of consecutive vertices being adjacent. A path will be called closed if its first vertex is adjacent to the last one. By a chord of a path $P$ in a graph $G$ we mean an edge of $G$ joining two nonconsecutive vertices of $P$. If $e$ is a chord in a closed path $P$, then $e$ is called proper if it is not the edge joining the first vertex of $P$ to its last vertex. Note that a proper chord of a closed path corresponds to the standard notion of a chord in a cycle. A snake in a graph $G$ is a closed path in $G$ without proper chords, and an open snake in $G$ is a path in $G$ without chords.

If $G$ and $H$ are graphs, then the product of $G$ and $H$ is the graph $G \times H$ with $V(G) \times V(H)$ as the vertex set and $(g_1, h_1)$ adjacent to $(g_2, h_2)$ if either $g_1g_2 \in E(G)$ and $h_1 = h_2$, or else $g_1 = g_2$ and $h_1h_2 \in E(H)$. Let $K_n^d$ be the product of $d$ copies of the complete graph $K_n$, $n \geq 2$, $d \geq 1$. It will be convenient to think of the vertices of $K_n^d$ as $d$-tuples of $n$-ary digits, i.e. the elements of the set $\{0, 1, \ldots, n-1\}$, with edges between two $d$-tuples differing at exactly one coordinate.

Let $S(K_n^d)$ be the length of the longest snake in $K_n^d$. The problem of estimating the value of $S(K_2^d)$ (known also as the snake-in-the-box problem) has an extensive literature. See [2], [6], [8], and the references in these papers. Evdokimov [6] was the first to prove that for some constant $c > 0$,

$$S(K_2^d) \geq c2^d. \quad (1)$$

The constant $c$ he got in his proof is very small ($c = 2^{-11}$). Other shorter proofs of (1) and with larger values of the constant $c$ have been given by Abbott and Katchalski [2] and Wojciechowski [8]. The best lower bound for $S(K_2^d)$ has been given by Abbott and Katchalski [4]. They show that (1) holds with $c = \frac{57}{256} = 0.300781 \ldots$. In [9] Wojciechowski proves a result of a different but related nature, namely, that for every $d \geq 2$, $K_2^d$ can be decomposed into 16 vertex disjoint snakes.
When $n \geq 3$, the existence of snakes of length $cn^d$, where $c$ is a positive constant independent of $d$, is known only for $n$ even. Abbott and Katchalski [5] proved that if $n \equiv 0 \mod 4$, then for $d \geq 3$

$$S(K^d_n) \geq \left(\frac{n}{2}\right)^{d-1} S(K^d_2).$$

(2)

It follows from (1) and (2) that in the case when $n \equiv 0 \mod 4$ the following theorem is true.

**Theorem 1.** For any even integer $n \geq 2$, there exists a constant $c_n > 0$ such that

$$S(K^d_n) \geq c_n n^d,$$

(3)

for every $d \geq 2$.

When $n \equiv 2 \mod 4$, the validity of Theorem 1 follows from the remark in [5] that the construction used in the proof of (2) can be modified to give long snakes in the case when $n \equiv 2 \mod 4$.

For $n$ odd, the best known lower bounds on $S(K^d_n)$ are not as good as for even $n$. Abbott and Dierker [1] proved that

$$S(K^d_n) \geq 4 \left\lfloor \frac{n}{2} \right\rfloor^{d-1} + 2 \sum_{k=1}^{d-2} \left\lfloor \frac{n}{2} \right\rfloor^k$$

for $n \geq 2$, $d \geq 2$,

and

$$S(K^d_n) \geq \begin{cases} 2n^{d-\ell-1} & \text{if } 2^\ell < d < 2^{\ell+1} \\ 2n^{d-\ell} & \text{if } d = 2^{\ell}. \end{cases}$$

It follows that for every $n \geq 2$ there is a constant $\lambda_n > 0$ such that

$$S(K^d_n) \geq \frac{1}{d^{\lambda_n}} n^d.$$

Abbott and Katchalski [5] ask whether there exists a constant $c_3 > 0$ such that (3) holds for $n = 3$. In this paper we give a positive answer to this question, and what is more, we show that the following general result is true.
Theorem 2. For any odd integer \( n \geq 3 \), there exists a constant \( c_n > 0 \) such that

\[
S(K_n^d) \geq c_n n^d,
\]

for every \( d \geq 2 \).

Actually, we prove a stronger result.

Theorem 3. For any odd integer \( n \geq 3 \), and any \( d \geq 5 \)

\[
S(K_n^d) \geq 2(n - 1)n^{d-4}.
\]

Thus, in particular, for \( n = 3 \) we get

\[
S(K_3^d) \geq \frac{4}{81}3^d.
\]

The proof of Theorem 3 will be given in section 4.

2. Basic Definitions

An \( m \)-path in a graph is a path containing \( m \) vertices, i.e. a path of length \( m - 1 \). If \( P \) is an \( m \)-path, then we will write \( m = |P| \). A chain \( C \) of paths in a graph \( G \) is a sequence \((P_1, P_2, \ldots, P_m)\) of paths in \( G \) such that each path in \( C \) has at least two vertices, and the last vertex of \( P_i \) is equal to the first vertex of \( P_{i+1} \), \( i = 1, 2, \ldots, m - 1 \). When the number \( m \) of paths in a chain needs to be specified, we shall refer to an \( m \)-chain of paths. An \( m \)-chain \( C = (P_i)_{i=1}^m \) of paths will be called closed if the first vertex of \( P_1 \) is equal to the last vertex of \( P_m \).

Now we are going to define an important operation that will be used throughout the paper. Given an \( m \)-path \( P = (g_i)_{i=1}^m \) in a graph \( G \) and an \( m \)-chain of paths \( C = (P_i)_{i=1}^m \) in a graph \( H \), let \( P \otimes C \) be the \( \left( \sum_{i=1}^m |P_i| \right) \)-path in the graph \( G \times H \) constructed in the following way. For each path \( P_i = (h_{i1}, h_{i2}, \ldots, h_{ik_i}) \) in \( C \) let \( P'_i \) be the path
\((g_{i}, h_{i1}), (g_{i}, h_{i2}), \ldots, (g_{i}, h_{ik})\) in \(G \times H\). Note that for each \(i \in \{1, 2, \ldots, m - 1\}\) the last vertex of the path \(P'_i\) is adjacent to the first vertex of the path \(P'_{i+1}\). Let \(P \otimes C\) be the path obtained by joining together (juxtaposing) the paths \(P'_1, P'_2, \ldots, P'_m\). We will say that \(P \otimes C\) is the path generated by \(P\) and \(C\). Note that the path generated by a closed path and a closed chain of paths is a closed path.

If \(D\) is a \(km\)-chain of paths in a graph \(H\), then the \(m\)-splitting of \(D\) is the sequence \((D_1, D_2, \ldots, D_m)\) of \(k\)-chains of paths in \(H\) which joined together (juxtaposed) give \(D\). The above definition of the operation \(\otimes\) can be generalized to the following situation. Let \(C = (P_i)_{i=1}^m\) be an \(m\)-chain of \(k\)-paths in a graph \(G\), let \(D\) be a \(km\)-chain of paths in \(H\), and let \((D_1, D_2, \ldots, D_m)\) be the \(m\)-splitting of \(D\). Note that for each \(i \in \{1, 2, \ldots, m - 1\}\) the last vertex of the path \(P_i \otimes D_i\) in the graph \(G \times H\) is equal to the first vertex of the path \(P_{i+1} \otimes D_{i+1}\). Set
\[C \otimes D = (P_1 \otimes D_1, P_2 \otimes D_2, \ldots, P_m \otimes D_m).\]

We will say that \(C \otimes D\) is the chain of paths generated by \(C\) and \(D\). Note that the chain of paths generated by two closed chains of paths is also a closed chain of paths. It is straightforward to verify that the operation \(\otimes\) is associative in the following sense.

**Property 1.** If \(P\) is an \(m\)-path in a graph \(G\), \(C\) is an \(m\)-chain of \(k\)-paths in a graph \(H\), and \(D\) is a \(km\)-chain of paths in a graph \(K\), then
\[(P \otimes C) \otimes D = P \otimes (C \otimes D).\]

Let \(C = (P_i)_{i=1}^m\) be a chain of paths in a graph \(G\). We say that \(C\) is openly separated if for \(i \leq m - 1\) and \(j = i + 1\), \(P_i\) and \(P_j\) have exactly one vertex in common, and otherwise \(P_i\) and \(P_j\) are vertex disjoint. We say that \(C\) is closely separated if \(C\) is closed, \(P_i\) and \(P_j\) have exactly one vertex in common when either \(i \leq m - 1\) and \(j = i + 1\), or \(i = 1\) and \(j = m\),
and otherwise $P_i$ and $P_j$ are vertex disjoint. The following property is a straightforward consequence of the definitions and its proof is left to the reader as an exercise.

**Property 2.**

(i) The path $P \otimes C$ generated by a path $P$ in a graph $G$ and an openly separated chain $C$ of open snakes in a graph $H$ is an open snake in the graph $G \times H$.

(ii) The closed path $P \otimes C$ generated by a closed path $P$ in a graph $G$ and a closely separated chain $C$ of open snakes in a graph $H$ is a snake in the graph $G \times H$. □

If $P$ is a path, then let $-P$ be the path obtained from $P$ by reversing the order of vertices, and if $C = (P_i)_{i=1}^{m}$ is a chain of paths, then let $-C = (-P_m, -P_{m-1}, \ldots, -P_1)$ be the chain of paths obtained from $C$ by reversing the order of paths and reversing every path.

The expression $(-1)^i X$, where $X$ is a path or a chain of paths, will mean $X$ for $i$ even and $-X$ for $i$ odd. Obviously, the following property is true.

**Property 3.** If $P$ is an $m$-path in a graph $G$ and $C$ is an $m$-chain of paths in a graph $H$, then $(-P) \otimes C = -(P \otimes -C)$. □

Let $C$ be a $km$-chain of paths, and let $S = (C_1, C_2, \ldots, C_m)$ be the $m$-splitting of $C$. By the alternate matrix of the splitting $S$ we mean the following $(m \times k)$-matrix $A$ of paths:

$$A = \begin{pmatrix} C_1 \\ -C_2 \\ \vdots \\ (-1)^{m-1}C_m \end{pmatrix} = \begin{pmatrix} Q_1^1 & Q_1^2 & \cdots & Q_1^k \\ Q_2^1 & Q_2^2 & \cdots & Q_2^k \\ \vdots & \vdots & \ddots & \vdots \\ Q_m^1 & Q_m^2 & \cdots & Q_m^k \end{pmatrix}$$

where $(Q_1^1, Q_2^1, \ldots, Q_k^1)$ is the sequence of paths forming the $k$-chain $(-1)^{i-1}C_i$. The splitting $S$ will be called **openly alternating** if for every odd $j$, $1 \leq j \leq m - 1$, the paths $Q_j^k$ and $Q_j^{k+1}$ have exactly one vertex in common, for every even $j$, $2 \leq j \leq m - 1$, the paths $Q_j^1$ and $Q_j^{1+1}$ have exactly one vertex in common, and otherwise the paths $Q_j^j$ and $Q_j^\ell$ are vertex disjoint, $i \in \{1, 2, \ldots, k\}$, $j, \ell \in \{1, 2, \ldots, m\}$, $j \neq \ell$. Note that the splitting $S$ is openly alternating.
if for every column of its alternate matrix $\mathcal{A}$ the paths in the column are mutually vertex disjoint except for the shared vertices which are necessary for $C$ to be a chain of paths, i.e. $Q^k_1$ and $Q^k_2$ have exactly one vertex in common, $Q^1_1$ and $Q^1_2$ have exactly one vertex in common, and so on.

Assume now that the $km$-chain $C$ is a closed chain of paths and $m$ is even. Then, we say that the splitting $S$ is closely alternating if for every odd $j$, $1 \leq j \leq m - 1$, the paths $Q^i_j$ and $Q^i_{j+1}$ have exactly one vertex in common, for every even $j$, $2 \leq j \leq m - 1$, the paths $Q^i_1$ and $Q^i_{j+1}$ have exactly one vertex in common, the paths $Q^i_1$ and $Q^i_m$ have exactly one vertex in common, and otherwise the paths $Q^i_j$ and $Q^i_\ell$ are vertex disjoint, $i \in \{1, 2, \ldots, k\}$, $j, \ell \in \{1, 2, \ldots, m\}$, $j \neq \ell$. As above, note that the splitting $S$ is closely alternating if for every column of its alternate matrix $\mathcal{A}$ the paths in the column are mutually vertex disjoint except for the shared vertices which are necessary for $C$ to be a closed chain of paths. The following property is a straightforward consequence of the definitions.

**Property 4.** Let $P$ be a $k$-path in a graph $G$, and let $D$ be a $km$-chain of paths in a graph $H$.

(i) If $C$ is the $m$-chain $(P, -P, \ldots, (-1)^{m-1}P)$, and the $m$-splitting of $D$ is openly alternating, then the $m$-chain $C \otimes D$ of paths in the graph $G \times H$ is openly separated.

(ii) If $m$ is even, $C$ is the closed $m$-chain $(P, -P, P, -P, \ldots, -P)$, and the $m$-splitting of $D$ is closely alternating, then the closed $m$-chain $C \otimes D$ of paths in the graph $G \times H$ is closely separated. \hfill \Box

Assume that $n \geq 3$ is a fixed odd integer. For every integer $d \geq 1$, we are going now to define the $n^d$-path $\pi^d_n$ in $K^d_n$, and the closed $(n - 1)n^d$-path $\gamma^{d+1}_n$ in $K^{d+1}_n$. These paths will be used in the construction of long snakes.

Let $\pi^1_n$ be the $n$-path $(0, 1, \ldots, n - 1)$ in $K_n$, and let $\gamma_n$ be the closed $(n - 1)$-path
(0,1,\ldots,n-2) in K_n. If d \geq 1 and the path \pi_n^d in K_n^d is defined, then let
\[ \pi_n^{d+1} = \pi_n^1 \otimes (\pi_n^d, -\pi_n^d, \pi_n^d, \ldots, \pi_n^d) \]
and
\[ \gamma_n^{d+1} = \gamma_n \otimes (\pi_n^d, -\pi_n^d, \pi_n^d, -\pi_n^d, \ldots, -\pi_n^d). \]

Let H be a graph, d \geq 1 be an integer, C be an n^d\text{-chain} of paths in H, and D be an (n-1)n^d\text{-chain} of paths in H. We say that C is \textit{openly well distributed} if either \( d = 1 \) and C is an openly separated chain of open snakes, or \( d \geq 2 \), every chain \( \mathcal{C}_i \) in the n-splitting \( \mathcal{S} = (\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \) of C is openly well distributed and \( \mathcal{S} \) is openly alternating. We also say that D is \textit{closely well distributed} if every chain \( \mathcal{D}_i \) in the (n-1)-splitting \( \mathcal{S}' = (\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{n-1}) \) of D is openly well distributed and \( \mathcal{S}' \) is closely alternating. The following property can be proved by a straightforward induction with respect to d.

\textbf{Property 5.} If C is an openly well distributed n^d\text{-chain} of paths in a graph H, then the chain \(-C\) is also openly well distributed. \( \square \)

Now we are ready to prove the following important lemma.

\textbf{Lemma 1.} If \( d \geq 1 \) and C is an openly well distributed n^d\text{-chain} of paths in a graph H, then the path \( \pi_n^d \otimes C \) is an open snake in the graph \( K_n^d \times H \).

\textit{Proof.} We are going to use induction with respect to d. For \( d = 1 \), the lemma is true by Property 2 (i). Assume that \( d \geq 1 \), and that C is an openly well distributed n^{d+1}\text{-chain} of paths in the graph H. Let \( \mathcal{S} = (\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n) \) be the n-splitting of C. By the definition, we have \( \pi_n^{d+1} \otimes C = (\pi_n^1 \otimes \mathcal{D}) \otimes C \), where \( \mathcal{D} = (\pi_n^d, -\pi_n^d, \ldots, \pi_n^d) \). By Property 1, the chain \( \pi_n^{d+1} \otimes C \) is equal to \( \pi_n^1 \otimes (\mathcal{D} \otimes C) \). We have
\[ \mathcal{D} \otimes C = (\pi_n^1 \otimes \mathcal{C}_1, (\pi_n^1) \otimes \mathcal{D}_2, \ldots, \pi_n^1 \otimes \mathcal{C}_n) \]
8
By Property 3,
\[ D \odot C = \left( \pi_n^d \odot C_1, -\left( \pi_n^d \odot -C_2 \right), \ldots, \pi_n^d \odot C_n \right) \]
Since the chain \( C \) is openly well distributed, the chains \( C_1, C_2, \ldots, C_n \) are also openly well distributed. By Property 5, the chains \( C_1, -C_2, \ldots, C_n \) are openly well distributed, so by the inductive hypothesis the paths \( \pi_n^d \odot C_1, \pi_n^d \odot -C_2, \ldots, \pi_n^d \odot C_n \) are open snakes in \( K_{d}^n \times H \). Hence \( D \odot C \) is a chain of open snakes in \( K_{d}^n \times H \). The splitting \( S \) is openly alternating, so by Property 4 (i), the chain \( D \odot C \) is openly separated. Hence by Property 2 (ii), \( \pi_1^n \odot (D \odot C) \) is an open snake in \( K_n \times K_{d}^n \times H = K_{d+1}^n \times H \), and the proof is complete.

The following lemma will be used in the proof of Theorem 3.

**Lemma 2.** If \( d \geq 1 \) and \( C \) is a closely well distributed \((n-1)n^d\)-chain of paths in a graph \( H \), then the path \( \gamma_n^{d+1} \odot C \) is a snake in the graph \( K_{d+1}^n \times H \).

**Proof.** Assume that \( d \geq 1 \) and that \( C \) is a closely well distributed \((n-1)n^d\)-chain of paths in the graph \( H \). Let \( S = (C_1, C_2, \ldots, C_{n-1}) \) be the \((n-1)\)-splitting of \( C \). By the definition we have \( \gamma_n^{d+1} \odot C = (\gamma_n \odot D) \odot C \), where \( D \) is the \((n-1)\)-chain \((\pi_n^d, -\pi_n^d, \ldots, -\pi_n^d)\). By Property 1, the chain \( \gamma_n^{d+1} \odot C \) is equal to \( \gamma_n \odot (D \odot C) \). We have
\[
D \odot C = \left( \pi_n^d \odot C_1, (\pi_n^d \odot -C_2), \ldots, (-\pi_n^d) \odot C_{n-1} \right)
\]
By Property 3,
\[
D \odot C = \left( \pi_n^d \odot C_1, (\pi_n^d \odot -C_2), \ldots, (-\pi_n^d \odot -C_{n-1}) \right)
\]
By Property 5 and Lemma 1, \( D \odot C \) is a chain of open snakes in \( K_{d}^n \). The splitting \( S \) is closely alternating so by Property 4 (ii), the chain \( D \odot C \) is closely separated. Hence by Property 2 (ii), \( \gamma_n \odot (D \odot C) \) is a snake in \( K_n \times K_{d}^n \times H = K_{d+1}^n \times H \), and the proof is complete. \( \square \)
3. Construction of Well Distributed Chains

Let $C$ be a chain of paths in a graph $G$. We say that $C$ joins $u_1$ to $u_2$ if $u_1$ is the first vertex of the first path of $C$ and $u_2$ is the last vertex of the last path. If $v$ is the first or the last vertex of a path $P$, then let $P - v$ be the path obtained by removing $v$ from $P$. Let $C = (P_i)_{i=1}^m$ be a chain of paths joining $u_1$ to $u_2$, and $D = (Q_i)_{i=1}^m$ be a chain of paths joining $v_1$ to $v_2$. We say that $C$ and $D$ are parallel if for every $i$, $1 \leq i \leq m$, the paths $P_i$ and $Q_i$ are vertex disjoint.

We also say that $C$ and $D$ are internally parallel if the following conditions are satisfied:

(i) the paths $P_1 - u_1$ and $Q_1$ are vertex disjoint,

(ii) the paths $P_1$ and $Q_1 - v_1$ are vertex disjoint,

(iii) the paths $P_m - u_2$ and $Q_m$ are vertex disjoint,

(iv) the paths $P_m$ and $Q_m - v_2$ are vertex disjoint, and

(v) for every $i$, $1 < i < m$, the paths $P_i$ and $Q_i$ are vertex disjoint.

Let $n \geq 2$ be an integer and let $H$ be a graph. An $n$-net in $H$ is a pair $(U, M)$, where $U = \{u_1, u_2, \ldots, u_{n+3}\}$ is a set of $n + 3$ vertices of $H$ and

$$M = \{N_{s,t}^i : i \in \{0, 1, \ldots, n - 1\}; s, t \in \{1, 2, \ldots, n + 3\}; s \neq t\}$$

is a set of openly separated chains of open snakes in $H$ such that $N_{s,t}^i$ joins $u_s$ to $u_t$, $N_{s,t}^i = -N_{t,s}^i$, and if $i \neq j$ then $N_{s,t}^i$ and $N_{v,w}^j$ are internally parallel, for any $i, j \in \{0, 1, \ldots, n - 1\}$, and $s, t, v, w \in \{1, 2, \ldots, n + 3\}$, $s \neq t$, $v \neq w$.

Assume now that $n \geq 3$ is a fixed odd integer. Let $X_n$ be the following $n \times (n+3)$-matrix.

$$X_n = \begin{pmatrix}
5 & 6 & 3 & 4 & 7 & 8 & 9 & \ldots & n+3 & 1 & 2 \\
4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots & 1 & 2 & 3 \\
7 & 8 & 5 & 6 & 9 & 10 & 11 & \ldots & 2 & 3 & 4 \\
6 & 7 & 8 & 9 & 10 & 11 & 12 & \ldots & 3 & 4 & 5 \\
9 & 10 & 7 & 8 & 11 & 12 & 13 & \ldots & 4 & 5 & 6 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\
n+2 & n+3 & n & n+1 & 1 & 2 & 3 & \ldots & n-3 & n-2 & n-1 \\
n+1 & n+2 & n+3 & 1 & 2 & 3 & 4 & \ldots & n-2 & n-1 & n \\
1 & 2 & n+2 & n+3 & 3 & 4 & 5 & \ldots & n-1 & n & n+1
\end{pmatrix}$$

10
Let $Y_n$ be the following $(n - 1) \times (n + 3)$-matrix.


divide

The matrix $X_n$ can be obtained by taking as rows the $n$ consecutive cyclic permutations of the sequence $(3, 4, \ldots, n + 3, 1, 2)$, and then for each odd row exchanging the entries at the positions 1 and 3 and exchanging the entries at the positions 2 and 4. The matrix $Y_n$ can be obtained from $X_n$ by removing the last row and modifying the $n - 1$ row.

From now on, to the end of this section, let us assume that we are given a graph $H$, and an $n$-net $(U, \mathcal{M})$ in $H$. Let $C$ be an $n^d$-chain of paths in $H$, $d \geq 1$. If every $n$-chain in the $n^{d-1}$-splitting of $C$ belongs to $\mathcal{M}$, then we say that $C$ is $\mathcal{M}$-built. For each $i = 1, 2, \ldots, n$, let $\varphi_i : \mathcal{M} \to \mathcal{M}$ be defined by

$$
\varphi_i(N_{i,v,w}^j) = N_{i+x_i,v,x_i+w}^{j+i-1 \mod n},
$$

where the bottom indices are taken from the matrix $X_n = (x_{p,r})$. Analogously, for each $i = 1, 2, \ldots, n - 1$, let $\psi_i : \mathcal{M} \to \mathcal{M}$ be defined by

$$
\psi_i(N_{i,v,w}^j) = N_{y_i+v,y_i+w}^{j+i-1 \mod n},
$$

where the bottom indices are taken from the matrix $Y_n = (y_{p,r})$. If $C$ is any $\mathcal{M}$-built $n^d$-chain, then $\varphi_i(C)$ and $\psi_i(C)$ are the $n^d$-chains obtained by applying $\varphi_i$ and $\psi_i$, respectively, to each of the $n$-chains in the $n^{d-1}$-splitting of $C$.

Let $D_1$ and $D_2$ be $\mathcal{M}$-built $n^d$-chains with $n^{d-1}$-splittings $(N_{s_k,t_k}^{i_k})_{k=1}^{n^{d-1}}$ and $(N_{v_k,w_k}^{j_k})_{k=1}^{n^{d-1}}$ respectively. If $i_k = j_k$ for $1 \leq k \leq n^{d-1}$, $s_k = v_k$ for $1 < k \leq n^{d-1}$, and $t_k = w_k$ for $1 \leq k < n^{d-1}$, then we say that $D_1$ and $D_2$ are internally $\mathcal{M}$-compatible. If $i_k \neq j_k$ for $1 \leq k \leq n^{d-1}$, $s_k \neq v_k$ for $1 < k \leq n^{d-1}$, and $t_k \neq w_k$ for $1 \leq k < n^{d-1}$, then we say that
$D_1$ and $D_2$ are internally $M$-parallel. If $i_k \neq j_k$ for $1 \leq k \leq n^{d-1}$, $s_k \neq v_k$ for $1 \leq k \leq n^{d-1}$, and $t_k \neq w_k$ for $1 \leq k \leq n^{d-1}$, then we say that $D_1$ and $D_2$ are $M$-parallel.

Let $\mathcal{C}$ be an $M$-built $n^{d+1}$-chain, let $\mathcal{S} = (C_1, C_2, \ldots, C_n)$ be the $n$-splitting of $\mathcal{C}$, and let

$$A = \begin{pmatrix}
C_1 \\
-\bar{C}_2 \\
C_3 \\
\vdots \\
-\bar{C}_{n-1} \\
\bar{C}_n
\end{pmatrix}$$

be the alternate matrix of $\mathcal{S}$. We say that $\mathcal{C}$ is $M$-well distributed if either $d = 0$, or $d > 0$ and the following conditions are satisfied.

(i) For every $i$, $1 \leq i \leq n$, the $n^d$-chain $C_i$ is $M$-well distributed.

(ii) Any two consecutive rows of $A$ form a pair of internally $M$-parallel chains.

(iii) Any two nonconsecutive rows of $A$ form a pair of $M$-parallel chains.

Since the $n$-splitting of an $M$-well distributed chain is openly alternating, the following property can be proved by induction with respect to $d$.

**Property 6.** If $\mathcal{C}$ is an $M$-well distributed $n^d$-chain, then $\mathcal{C}$ is an openly well distributed chain.

Since no integer appears twice in any row of $X_n$, no integer appears twice in any row of $Y_n$, and $N^i_{s,t} = -N^i_{t,s}$, a straightforward induction with respect to $d$ can be used to prove the following property.

**Property 7.** If $\mathcal{C}$ is an $M$-well distributed $n^d$-chain, then for any $i \in \{1, 2, \ldots, n\}$ and any $j \in \{1, 2, \ldots, n-1\}$ the chains $\varphi_i(\mathcal{C})$, $\psi_j(\mathcal{C})$, and $-\mathcal{C}$ are also $M$-well distributed.

Now we are ready to prove the following important lemma.

**Lemma 3.** For any $d \geq 1$ there exist four internally $M$-compatible $n^d$-chains $C^d_{3,1}$, $C^d_{3,2}$, $C^d_{4,1}$
and \( C_{4,2}^d \), such that \( C_{s,t}^d \) is \( \mathcal{M} \)-well distributed and joins \( u_s \) to \( u_t \), for any \( s \in \{3, 4\} \), \( t \in \{1, 2\} \).

**Proof.** We shall use induction with respect to \( d \). If \( d = 1 \), then let \( C_{s,t}^1 = \mathcal{N}_{s,t}^0 \), for each \( s \in \{3, 4\} \), and each \( t \in \{1, 2\} \). Clearly, the \( n \)-chains \( C_{3,1}^1, C_{3,2}^1, C_{4,1}^1 \) and \( C_{4,2}^1 \) have the required properties.

Suppose now that \( d \geq 1 \) and the \( n' \)-chains \( C_{3,1}^d, C_{3,2}^d, C_{4,1}^d \) and \( C_{4,2}^d \) satisfy the specified conditions. Given \( s \in \{3, 4\} \) and \( t \in \{1, 2\} \), let \( C_{s,t}^{d+1} \) be the \( n'' \)-chain with the \( n \)-splitting \( S = (C_1, C_2, \ldots, C_n) \) such that \( C_1 = \varphi_1(C_{s,t}^d), C_i = \varphi_i(C_{s,t}^d) \) for odd \( i, C_i = -\varphi_i(C_{s,t}^d) \) for even \( i, 1 < i < n \), and \( C_n = \varphi_n(C_{s,t}^d) \).

Let \( m = n'-1 \). Since \( C_{3,1}^d, C_{3,2}^d, C_{4,1}^d \), and \( C_{4,2}^d \) are internally \( \mathcal{M} \)-compatible, there are \( i_1, i_2, \ldots, i_m \in \{0, 1, \ldots, n-1\} \) and \( z_1, z_2, \ldots, z_{m-1} \in \{1, 2, \ldots, n+3\} \) such that for any \( v \in \{3, 4\} \) and \( w \in \{1, 2\} \), the sequence

\[
(\mathcal{N}_{i_1z_1}, \mathcal{N}_{i_2z_2}, \mathcal{N}_{i_3z_3}, \ldots, \mathcal{N}_{(i_m+1)z_{m-1}}, \mathcal{N}_{iz_{m-1}w})
\]

is the \( m \)-splitting of \( C_{v,w}^d \). Thus, if \( A \) is the alternate matrix of the splitting \( S \), then we have:

\[
A = \begin{pmatrix}
C_1 & \varphi_1(C_{3,1}^d) & \varphi_1(C_{3,2}^d) & \ldots & \varphi_1(C_{3,2m}^d) \\
-C_2 & \varphi_2(C_{3,1}^d) & \varphi_2(C_{3,2}^d) & \ldots & \varphi_2(C_{3,2m}^d) \\
-C_3 & \varphi_3(C_{3,1}^d) & \varphi_3(C_{3,2}^d) & \ldots & \varphi_3(C_{3,2m}^d) \\
-C_4 & \varphi_4(C_{3,1}^d) & \varphi_4(C_{3,2}^d) & \ldots & \varphi_4(C_{3,2m}^d) \\
-C_5 & \varphi_5(C_{3,1}^d) & \varphi_5(C_{3,2}^d) & \ldots & \varphi_5(C_{3,2m}^d) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-C_{n-1} & \varphi_{n-1}(C_{3,1}^d) & \varphi_{n-1}(C_{3,2}^d) & \ldots & \varphi_{n-1}(C_{3,2m}^d) \\
C_n & \varphi_n(C_{3,1}^d) & \varphi_n(C_{3,2}^d) & \ldots & \varphi_n(C_{3,2m}^d)
\end{pmatrix}
\]

We will prove now that \( C_{s,t}^{d+1} \) satisfies the required conditions. To verify that \( C_{s,t}^{d+1} \) is a chain of paths, note that the entries of the matrix \( X_n = (x_{j,k}) \) satisfy \( x_{i,1} = x_{i+1,2} \) for \( i \) odd, and \( x_{i,3} = x_{i+1,4} \) for \( i \) even, \( 1 \leq i \leq n-1 \). Since \( x_{1,s} = s \) and \( x_{n,t} = t \), the chain \( C_{s,t}^{d+1} \) joins \( u_s \) to \( u_t \). Since the chains \( C_{v,w}^d \) are \( \mathcal{M} \)-well distributed, Property 7 implies that the chains \( C_1, C_2, \ldots, C_n \) are also \( \mathcal{M} \)-well distributed. Since the integers 3 and 4 do not appear in the 3rd or 4th column of the matrix \( X_n \) except in the first row, the integers 1 and 2 do.
not appear in the 1st or 2nd column of $X_n$ except in the last row, and no integer appears twice in any column of $X_n$, it follows from the definition of the functions $\varphi_i$ that any two consecutive rows of $A$ are internally $\mathcal{M}$-parallel chains and any two nonconsecutive rows of $A$ are $\mathcal{M}$-parallel chains. Hence the chain $C_{d,t}^{d+1}$ is $\mathcal{M}$-well distributed. Since it is clear that the chains $C_{3,1}^{d+1}, C_{3,2}^{d+1}, C_{4,1}^{d+1}$ and $C_{4,2}^{d+1}$ are internally $\mathcal{M}$-compatible, the proof is complete.  

The following lemma will be used in the proof of Theorem 3.

**Lemma 4.** For any $d \geq 1$ there exists a closely well distributed $\mathcal{M}$-built $(n-1)n^d$-chain.

**Proof.** Let $d \geq 1$. Let $D_n^{d+1}$ be the closed $(n-1)n^d$-chain with the $(n-1)$-splitting $S = (C_1, C_2, \ldots, C_{n-1})$ such that $C_i = \psi_i(C_{i+1}^d)$ for odd $i$, and $C_i = -\psi_i(C_{i+1}^d)$ for even $i, 1 \leq i \leq n-1$. Thus, if $A$ is the alternate matrix of the splitting $S$, and $m = n^{d-1}$, then we have:

\[
A = \begin{pmatrix}
C_1 & \psi_1(C_{4,1}^d) & \psi_1(C_{3,1}^d) & \psi_1(C_{2,1}^d) & \psi_1(C_{1,1}^d) \\
-C_2 & \psi_2(C_{4,1}^d) & \psi_2(C_{3,1}^d) & \psi_2(C_{2,1}^d) & \psi_2(C_{1,1}^d) \\
C_3 & \psi_3(C_{4,1}^d) & \psi_3(C_{3,1}^d) & \psi_3(C_{2,1}^d) & \psi_3(C_{1,1}^d) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
C_{n-2} & \psi_{n-2}(C_{4,1}^d) & \psi_{n-2}(C_{3,1}^d) & \psi_{n-2}(C_{2,1}^d) & \psi_{n-2}(C_{1,1}^d) \\
-C_{n-1} & \psi_{n-1}(C_{4,1}^d) & \psi_{n-1}(C_{3,1}^d) & \psi_{n-1}(C_{2,1}^d) & \psi_{n-1}(C_{1,1}^d)
\end{pmatrix}
\]

where, for some $i_1, i_2, \ldots, i_m \in \{0, 1, \ldots, n-1\}$, and $z_1, z_2, \ldots, z_m \in \{1, 2, \ldots, n+3\}$, the sequence

\[
(N_{i_1}^{z_1}, N_{i_2}^{z_2}, N_{i_3}^{z_3}, \ldots, N_{i_m}^{z_{m-2}}, N_{i_m-1}^{z_{m-1}}),
\]

is the $m$-splitting of the chain $C_{4,1}^d$, and the sequence

\[
(N_{i_1}^{z_1}, N_{i_2}^{z_2}, N_{i_3}^{z_3}, \ldots, N_{i_m}^{z_{m-1}}, N_{i_{m-1}}^{z_m}),
\]

is the splitting of the chain $C_{3,2}^d$.

Since the entries of the matrix $Y_n = (y_{j,k})$ satisfy $y_{i,1} = y_{i+1,2}$ for $i$ odd, $y_{i,3} = y_{i+1,4}$ for $i$ even, $1 \leq i \leq n-2$, and $y_{n-1,3} = y_{1,4}$, it follows that the sequence $D_n^{d+1}$ of paths is a closed chain of paths. Since the chains $C_{v,w}^d$ are $\mathcal{M}$-well distributed, Property 7 implies that
4. Construction of an $n$-Net in $K_n^3$

Let $n \geq 3$ be a fixed odd integer. The vertices of the graph $K_n^3$ are 3-tuples of digits from the set $\{0, 1, \ldots, n-1\}$. Given a vertex $v = (a_1, a_2, a_3)$ of $K_n^3$, we say that $a_i$ appears at the $i$-th position of $v$, $i = 1, 2, 3$. If $a \in \{0, 1, \ldots, n-1\}$, then let $\overline{a} = a + \frac{n+1}{2} \mod n$, and $\overline{a} = a + \frac{n+1}{2} \mod n$. If $a_1, a_2 \in \{0, 1, \ldots, n-1\}$, then let $[a_1, a_2] = \{a_1, a_1+1, a_1+2, \ldots, a_2\}$, where the addition is taken modulo $n$. Thus, for example, if $n = 7$ then $[5, 1] = \{5, 6, 0, 1\}$.

We are going now to define the $n$-net $(U_n, M_n)$ in $K_n^3$. Let

$$U_n = \{u_1, u_2, \ldots, u_{n+3}\} = \{(a_1, a_2, a_3) : a_1 \in \{0, 1, \ldots, \frac{n+1}{2}\}, \{a_2, a_3\} = \{\overline{a_1}, \overline{a_1}\}\}.$$

To construct the chains in the set $M_n$ we need to introduce some definitions. Suppose that $v = (a_1, a_2, a_3)$ is a vertex from the set $U_n$, and let us assume that the digit $b \notin \{a_1, a_2, a_3\}$. The adjunct of the digit $b$ in $v$ is the digit $\overline{a_1}$ if $b \in [a_1, \overline{a_1}]$, and the digit $\overline{a_1}$ otherwise. Let $\eta : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ be the function defined by $\eta(i) = i + 1$ if $1 \leq i < 3$, and $\eta(3) = 1$. For any $i \in \{0, 1, \ldots, n-1\}$ and $s \in \{1, 2, \ldots, n+3\}$, let $Q_i^s$ be the path starting with $u_s$ defined as follows. If the digit $i$ appears at the $j$-th position in $u_s$, then let $Q_i^s = (u_s, u_s')$, where $u_s'$ is obtained from $u_s$ by replacing the digit at the $\eta(j)$-th position with the digit $i$. If the digit $i$ does not appear in $u_s$, then let the adjunct of $i$ appear at the $j$-th position in $u_s$. Let $Q_i^s = (u_s, u_s', u_s'')$, where $u_s'$ is obtained from $u_s$ by replacing the digit at the $j$-th position with the digit $i$, and let $u_s''$ be obtained from $u_s'$ by replacing the digit at the $\eta(j)$-th position with the digit $i$. For example, if $n = 5$ and the sequence $u_1, u_2, \ldots, u_8$ of vertices in $U_5$ is

the chains $C_1, C_2, \ldots, C_{n-1}$ are also $M$-well distributed. By Property 6, $C_1, C_2, \ldots, C_{n-1}$ are openly well distributed. Since no integer appears twice in any columns of $Y_n$, it follows that any two consecutive rows of $A$ are internally $M$-parallel chains and any two nonconsecutive rows of $A$ are $M$-parallel chains, hence the splitting $S$ is closely alternating. Thus, $D_3^{s+1}$ is a closely well distributed $M$-built chain, and the proof is complete. \hfill \Box
where we write $a_1a_2a_3$ instead of $(a_1, a_2, a_3)$, then the path $Q^i_s$ is the path in the $s$-th row and the $(i + 1)$-st column of the following matrix:

$$
\begin{pmatrix}
(023,003) & (023,013,011) & (023,022) & (023,323) & (023,024,424) \\
(032,002) & (032,031,131) & (032,232) & (032,033) & (032,042,044) \\
(134,103,100) & (134,114) & (134,124,122) & (134,133) & (134,134) \\
(240,040) & (240,214,211) & (240,224) & (240,230,233) & (240,244) \\
(310,010) & (310,311) & (310,320,322) & (310,330) & (310,314,414)
\end{pmatrix}
$$

It can be verified that the following property is true.

**Property 8.** If $i, j \in \{0,1,\ldots,n-1\}$, $i \neq j$, and $s, t \in \{1,2,\ldots,n+3\}$, $s \neq t$, then the paths $Q^i_s$ and $Q^j_t$ have only the first vertex in common, and the paths $Q^i_s$ and $Q^j_t$ are vertex disjoint.  

Now we are ready to define the set

$$
\mathcal{M}_n = \left\{ \mathcal{N}^i_{s,t} : i \in \{0,1,\ldots,n-1\}; s, t \in \{1,2,\ldots,n+3\}; s \neq t \right\}
$$

of $n$-chains of paths in $K_n^3$. Given $i \in \{0,1,\ldots,n-1\}$, and $s, t \in \{1,2,\ldots,n+3\}$ such that $s < t$ let

$$
\mathcal{N}^i_{s,t} = (Q^i_s, P_1, P_2, \ldots, P_{n-2}, -Q^i_t),
$$

and

$$
\mathcal{N}^i_{t,s} = (Q^i_t, -P_{n-2}, -P_{n-3}, \ldots, -P_1, -Q^i_s),
$$

where $P_1, P_2, \ldots, P_{n-2}$ are defined as follows. Let $v_s, v_t$ be the last vertices of the paths $Q^i_s$ and $Q^i_t$ respectively, let $j_s, j_t \in \{1,2,3\}$ be such that $i$ does not appear neither at the $j_s$-th position of $v_s$, nor at the $j_t$-th position of $v_t$, and let $a_s, a_t$ be the digits at the position $j_s$ of $v_s$ and the position $j_t$ of $v_t$ respectively. Note that for $v \in \{v_s, v_t\}$, the digit $i$ appears at exactly two positions of $v$, hence $j_s$ and $j_t$ are well defined and $i \not\in \{a_s, a_t\}$. Now let us consider two cases.
If \( j_s = j_t \), then let \( b_2, b_3, \ldots, b_{n-2} \) be a sequence of different digits from the set \( \{0, 1, \ldots, n - 1\} \setminus \{a_s, a_t, i\} \), let \( b_1 = a_s \), \( b_{n-1} = a_t \), and let \( P_j \) be the path \((w_j, w_{j+1})\) in \( K_n^3 \), \( j = 1, 2, \ldots, n - 2 \), where \( w_k \) is obtained from \( v_s \) by replacing the digit at the \( i \)-th position with the digit \( b_k \), \( k = 1, 2, \ldots, n - 1 \). If \( j_s \neq j_t \), then let \( m = \frac{n-1}{2} \), let \( b_1 = a_s \), let \( b_2, b_3, \ldots, b_m \) be a sequence of different digits from the set \( \{0, 1, \ldots, n - 1\} \setminus \{a_s, i\} \), let \( b_{m+1}, b_{m+2}, \ldots, b_{2m-1} \) be a sequence of different digits from the set \( \{0, 1, \ldots, n - 1\} \setminus \{a_t, i\} \), and let \( b_{2m} = a_t \). For \( j \in \{1, 2, \ldots, 2m - 1\} \setminus \{m\} \) let \( P_j \) be the path \((w_j, w_{j+1})\) in \( K_n^3 \), and let \( P_m = (w_m, (i, i, i), w_{m+1}) \), where \( w_k \) is obtained from \( v_s \) by replacing the digit at the \( j_s \)-th position with the digit \( b_k \), for \( k = 1, 2, \ldots, m \), and \( w_k \) is obtained from \( v_t \) by replacing the digit at the \( j_t \)-th position with the digit \( b_k \), for \( k = m + 1, m + 2, \ldots, 2m \).

Assuming that in the construction of the paths \( P_1, P_2, \ldots, P_{n-2} \), we choose the sequences of digits to be increasing and containing as small digits as possible, in the case \( n = 5 \), with \( u_1, u_2, \ldots, u_8 \) as above, we have for example:

\[
\mathcal{N}_{1,2}^0 = \left( (023, 003), (001, 001), (001, 004), (004, 002), (002, 032) \right)
\]

\[
\mathcal{N}_{1,3}^0 = \left( (203, 003), (003, 001), (001, 000, 010), (010, 030), (030, 130, 134) \right)
\]

\[
\mathcal{N}_{3,4}^0 = \left( (134, 130, 030), (030, 010), (010, 000, 200), (200, 100), (100, 103, 143) \right)
\]

The following lemma will be used in the proof of Theorem 3.

**Lemma 5.** The pair \((U_n, \mathcal{M}_n)\) is an \( n \)-net in \( K_n^3 \).

**Proof.** Clearly, it follows from the construction and Property 8 that \( \mathcal{N}_{s,t}^i \) is an openly separated chain of open snakes joining \( u_s \) to \( u_t \), and \( \mathcal{N}_{s,t}^i = -\mathcal{N}_{s,t}^i \), for any \( i \in \{0, 1, \ldots, n - 1\} \) and \( s, t \in \{1, 2, \ldots, n+3\} \), \( s \neq t \). Note that for every path \( P \) of \( \mathcal{N}_{s,t}^i \) except the first and the last, the digit \( i \) appears at least at two positions in each vertex of \( P \). From the above fact and Property 8 it follows that \( \mathcal{N}_{s,t}^i \) and \( \mathcal{N}_{v,w}^j \) are internally parallel, for any \( i, j \in \{0, 1, \ldots, n - 1\} \) and \( s, t, v, w \in \{1, 2, \ldots, n+3\} \), such that \( i \neq j \) and \( v \neq w \), hence the proof is complete.

Now, we are ready to prove Theorem 3.

17
Proof of Theorem 3. We have $K^d_n = K^d_n \times K^3_n$ and $d - 3 \geq 2$. Let $(U_n, M_n)$ be the $n$-net in $K^3_n$ defined in this section (see Lemma 5). By Lemma 4, there is a closely well distributed $M_n$-built $(n - 1)n^{d-4}$-chain $D^{d-3}_n$. By Lemma 2, the closed path

\[ P = \gamma^{d-3}_n \otimes D^{d-3}_n \]

is a snake in $K^d_n$. Since

\[ |\gamma^{d-3}_n| = (n - 1)n^{d-4} \]

and each path in $D^{d-3}_n$ has length at least 2, we have

\[ S(K^d_n) \geq |P| \geq 2(n - 1)d^{n-4}, \]

and the proof is complete.

\[ \square \]

5. Concluding remarks

The construction of long snakes in $K^d_n$ presented in this paper can, after minor modifications, be used in the case when $n$ is even, and thus we can use it to prove Theorem 1. However, the value of the constant $c_n$ obtained in such a proof is not as good as when (1) and (2) are applied (see Introduction). It follows from (2) and (3) that there is a constant $c > 0$ independent of $n$ such that for any $n \equiv 0 \mod 4$, and any $d \geq 2$

\[ S(K^d_n) \geq cn^{d-1}. \]  

(4)

By the remark in [5], the constant $c$ can be taken in such a way that (4) holds for all even $n$.

A natural question arises.

Question 1. Is there a constant $c > 0$ such that

\[ S(K^d_n) \geq cn^{d-1}, \]

for every $n \geq 2$ and $d \geq 2$.

We would like to ask a stronger question.
Question 2. Is there a constant $c > 0$ and an integer $d_0$ such that for every $n \geq 2$ there is an $n$-net $(U_n^{d_0}, \mathcal{M}_n^{d_0})$ in the graph $K_n^{d_0}$, where every chain $C = (P_i)_{i=1}^n$ in the set $\mathcal{M}_n^{d_0}$ satisfies
\[
\sum_{i=1}^n |P_i| \geq cn^{d_0}.
\]

It follows from the proof of Theorem 3 that the positive answer to Question 2 implies that the answer to Question 1 is also positive.

The best known upper bound on $S(K_2^d)$ has been given by Snevily [7]
\[
S(K_2^d) \leq 2^{d-1} - \frac{2^{d-1}}{20d - 41}.
\]

In the general case, Abbott and Katchalski [3] proved that
\[
S(K_n^d) \leq \left(1 + \frac{1}{d-1}\right)n^{d-1}.
\]

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