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ORIGINAL PAPER

On the Sizes of Vertex-k-Maximal r-Uniform Hypergraphs

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Abstract

Let $H = (V, E)$ be a hypergraph, where V is a set of vertices and E is a set of non-empty subsets of V called edges. If all edges of H have the same cardinality r, then H is a r-uniform hypergraph; if E consists of all r-subsets of V, then H is a complete r-uniform hypergraph, denoted by K_n^r, where $n = |V|$. A hypergraph $H' = (V', E')$ is called a subhypergraph of $H = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. A r-uniform hypergraph $H = (V, E)$ is vertex-k-maximal if every subhypergraph of H has vertex-connectivity at most k, but for any edge $e \in E(K_n^r) \setminus E(H)$, $H + e$ contains at least one subhypergraph with vertex-connectivity at least $k + 1$. In this paper, we first prove that for given integers n, k, r with $k, r \geq 2$ and $n \geq k + 1$, every vertex-k-maximal r-uniform hypergraph H of order n satisfies $|E(H)| \geq \binom{n}{r} - \binom{n-k}{r}$, and this lower bound is best possible. Next, we conjecture that for sufficiently large n, every vertex-k-maximal r-uniform hypergraph H on n vertices satisfies $|E(H)| \leq \binom{n}{r} - \binom{n-k}{r} + (\frac{n}{k} - 2)\binom{k}{r}$, where $k, r \geq 2$ are integers. And the conjecture is verified for the case $r > k$.

Keywords Vertex-connectivity · Vertex-k-maximal hypergraphs · r-Uniform hypergraphs

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1 Introduction

In this paper, we consider finite simple graphs. For graph-theoretical terminolog and notation not defined here, we follow [4]. For a graph G, we use $\kappa(G)$ to den the *vertex-connectivity* of G. The *complement* of a graph G is denoted by G^c. I $X \subseteq E(G^c)$, $G+X$ is the graph with vertex set $V(G)$ and edge set $E(G) \cup X$. We v use $G+e$ for $G+\{e\}$. The *floor* of a real number x, denoted by $\lfloor x \rfloor$, is the great integer not larger than x; the *ceiling* of a real number x, denoted by $\lceil x \rceil$, is the le integer greater than or equal to x. For two integers n and k, we define $\binom{n}{k} = \frac{n!}{k!(n-}$ when $k \leq n$ and $\binom{n}{k} = 0$ when $k > n$.

Matula [14] first explicitly studied the quantity $\overline{\kappa}(G) = max\{\kappa(G') : G' \subseteq$ For a positive integer k, the graph G is *vertex-k-maximal* if $\overline{\kappa}(G) \leq k$ but for a edge $e \in E(G^c)$, $\overline{\kappa}(G+e) > k$. Because $\kappa(K_n) = n-1$, a vertex-k-maximal gra G with at most $k+1$ vertices must be a complete graph.

The *union* of two graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with ver set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *join* of two graphs G_1 and denoted by $G_1 \vee G_2$, is the graph obtained from the union of G_1 and G_2 by adding the edges that connect the vertices of G_1 with G_2. Let $G_{n,k} = ((p-1)K_k \cup K_q) \vee$ where $n = pk + q \geq 2k$ $(1 \leq q \leq k)$ and $(p-1)K_k$ is the union of $p-1$ compl graphs on k vertices. Then $G_{n,k}$ is vertex-k-maximal and $|E(G_{n,k})| \leq \frac{3}{2}(k - \frac{1}{3})(n-$ where the equality holds if n is a multiple of k. Mader [11] conjectured that, for la order of graphs, the graph $G_{n,k}$ would in fact present the best possible upper bou for the sizes of a vertex-k-maximal graph.

Conjecture 1 (Mader [11]) *Let $k \geq 2$ be an integer. Then for sufficiently large n, ev vertex-k-maximal graph on n vertices satisfies $|E(G)| \leq \frac{3}{2}(k - \frac{1}{3})(n-k)$.*

Some progresses towards Conjecture 1 are listed in the following.

Theorem 1.1 *Let $k \geq 2$ be an integer.*

(i) *(Mader [10], see also [11]) Conjecture 1 holds for $k \leq 6$.*
(ii) *(Mader [10], see also [11]) For sufficiently large n, every vertex-k-maximal gr G on n vertices satisfies $|E(G)| \leq (1 + \frac{1}{\sqrt{2}})k(n-k)$.*
(iii) *(Yuster [18]) If $n \geq \frac{9k}{4}$, then every vertex-k-maximal graph G on n verti satisfies $|E(G)| \leq \frac{193}{120}k(n-k)$.*
(iv) *(Bernshteyn and Kostochka [3]) If $n \geq \frac{5k}{2}$, then every vertex-k-maximal gr G on n vertices satisfies $|E(G)| \leq \frac{19}{12}k(n-k)$.*

In [17], Xu, Lai and Tian obtained the lower bound of the sizes of vertex-k-maxi graphs.

Theorem 1.2 (Xu, Lai and Tian [17]) *Let n, k be integers with $n \geq k+1 \geq 3$. If G vertex-k-maximal graph on n vertices, then $|E(G)| \geq (n-k)k + \frac{k(k-1)}{2}$. Furtherm this bound is best possible.*

The related studies on edge-k-maximal graphs have been conducted by quit few researchers, as seen in [7,9,12,13,15], among others. For corresponding digr problems, see [1,8], among others.

Let $H = (V, E)$ be a hypergraph, where V is a finite set and E is a set of non-empty subsets of V, called edges. An edge of cardinality 2 is just a graph edge. For a vertex $u \in V$ and an edge $e \in E$, we say u is *incident with* e or e is *incident with* u if $u \in e$. If all edges of H have the same cardinality r, then H is a *r-uniform hypergraph*; if E consists of all r-subsets of V, then H is a *complete r-uniform hypergraph*, denoted by K_n^r, where $n = |V|$. For $n < r$, the complete r-uniform hypergraph K_n^r is just the hypergraph with n vertices and no edges. The *complement* of a r-uniform hypergraph $H = (V, E)$, denoted by H^c, is the r-uniform hypergraph with vertex set V and edge set consisting of all r-subsets of V not in E. A hypergraph $H' = (V', E')$ is called a *subhypergraph* of $H = (V, E)$, denoted by $H' \subseteq H$, if $V' \subseteq V$ and $E' \subseteq E$. For $X \subseteq E(H^c)$, $H + X$ is the hypergraph with vertex set $V(H)$ and edge set $E(H) \cup X$; for $X' \subseteq E(H)$, $H - X'$ is the hypergraph with vertex set $V(H)$ and edge set $E(H) \setminus X'$. We use $H + e$ for $H + \{e\}$ and $H - e'$ for $H - \{e'\}$ when $e \in E(H^c)$ and $e' \in E(H)$. For $Y \subseteq V(H)$, we use $H[Y]$ to denote the hypergraph *induced* by Y, where $V(H[Y]) = Y$ and $E(H[Y]) = \{e \in E(H) : e \subseteq Y\}$. $H - Y$ is the hypergraph induced by $V(H) \setminus Y$.

Let H be a hypergraph and $V_1, V_2, \ldots, V_l$ be subsets of $V(H)$. An edge $e \in E(H)$ is *$(V_1, V_2, \ldots, V_l)$-crossing* if $e \cap V_i \neq \emptyset$ for $1 \leq i \leq l$. If in addition, $e \subseteq \cup_{i=1}^{l} V_i$, then e is *exact-$(V_1, V_2, \ldots, V_l)$-crossing*. The set of all $(V_1, V_2, \ldots, V_l)$-crossing edges of H is denoted by $E_H[V_1, V_2, \ldots, V_l]$; the set of all exact-$(V_1, V_2, \ldots, V_l)$-crossing edges of H is denoted by $E_{H[V_1 \cup V_2 \cup \cdots \cup V_l]}[V_1, V_2, \ldots, V_l]$. Let $d_H(V_1, V_2, \ldots, V_l) = |E_H[V_1, V_2, \ldots, V_l]|$ and $d_{H[V_1 \cup V_2 \cup \cdots \cup V_l]}(V_1, V_2, \ldots, V_l) = |E_{H[V_1 \cup V_2 \cup \cdots \cup V_l]}[V_1, V_2, \ldots, V_l]|$. For a vertex $u \in V(H)$, we call $d_H(u) := d_H(\{u\}, V(H) \setminus \{u\})$ the *degree* of u in H. The *minimum degree* $\delta(H)$ of H is defined as $min\{d_H(u) : u \in V\}$; the *maximum degree* $\Delta(H)$ of H is defined as $max\{d_H(u) : u \in V\}$. When $\delta(H) = \Delta(H) = k$, we call H *k-regular*.

Given a hypergraph H, we define a *walk* in H to be an alternating sequence $v_1, e_1, v_2, \ldots, e_s, v_{s+1}$ of vertices and edges of H such that: $v_i \in V(H)$ for $i = 1, \ldots, s+1$; $e_i \in E(H)$ for $i = 1, \ldots, s$; and $v_i, v_{i+1} \in e_i$ for $i = 1, \ldots, s$. A *path* is a walk with additional restrictions that the vertices are all distinct and the edges are all distinct. A hypergraph H is *connected* if for every pair of vertices $u, v \in V(H)$, there is a path connecting u and v; otherwise H is *disconnected*. A *component* of a hypergraph H is a maximal connected subhypergraph of H. A subset $X \subseteq V$ is called a *vertex-cut* of H if $H - X$ is disconnected. We define the *vertex-connectivity* of H, denoted by $\kappa(H)$, as follows: if H had at least one vertex-cut, then $\kappa(H)$ is the cardinality of a minimum vertex-cut of H; otherwise $\kappa(H) = |V(H)| - 1$. We call a hypergraph H *k-vertex-connected* if $\kappa(H) \geq k$. Let $\overline{\kappa}(H) = max\{\kappa(H') : H' \subseteq H\}$. For a positive integer k, the r-uniform hypergraph H is *vertex-k-maximal* if $\overline{\kappa}(H) \leq k$ but for any edge $e \in E(H^c)$, $\overline{\kappa}(H + e) > k$. Since $\kappa(K_n^r) = n - r + 1$, we note that H is complete if H is a vertex-k-maximal r-uniform hypergraph with $n - r + 1 \leq k$, where $n = |V(H)|$. The edge-k-maximal hypergraph can be defined similarly. For results on the connectivity of hypergraphs, see [2,5,6] for references.

In [16], we determined, for given integers n, k and r, the extremal sizes of an edge-k-maximal r-uniform hypergraph on n vertices.

Theorem 1.3 (Tian, Xu, Lai and Meng [16]) *Let k and r be integers with $k, r \geq 2$, and let $t = t(k, r)$ be the largest integer such that $\binom{t-1}{r-1} \leq k$. That is, t is the integer satisfying $\binom{t-1}{r-1} \leq k < \binom{t}{r-1}$. If H is an edge-k-maximal r-uniform hypergraph with $n = |V(H)| \geq t$, then*

(i) $|E(H)| \leq \binom{t}{r} + (n-t)k$, *and this bound is best possible;*
(ii) $|E(H)| \geq (n-1)k - ((t-1)k - \binom{t}{r})\lfloor \frac{n}{t} \rfloor$, *and this bound is best possible.*

The main goal of this research is to investigate, for given integers n, k and r, the extremal sizes of a vertex-k-maximal r-uniform hypergraph on n vertices. Section 2 below is devoted to the study of some properties of vertex-k-maximal r-uniform hypergraphs. In Sect. 3, we give the best possible lower bound of the sizes of vertex-k-maximal r-uniform hypergraphs. We propose a conjecture on the upper bound of the sizes of vertex-k-maximal r-uniform hypergraphs and verify the conjecture for the case $r > k$ in Sect. 4.

2 Properties of Vertex-k-Maximal r-Uniform Hypergraphs

Combining the definition of vertex-k-maximal r-uniform hypergraph with $\kappa(K_n^r) = n - r + 1$, we obtain that H is isomorphic to K_n^r if H is a vertex-k-maximal r-uniform hypergraph with $n = |V(H)| \leq k + r - 1$.

Lemma 2.1 *Let n, k, r be integers with $k, r \geq 2$ and $n \geq k + r - 1$. If H is a vertex k-maximal r-uniform hypergraph on n vertices, then $\overline{\kappa}(H) = \kappa(H) = k$.*

Proof Since H is vertex-k-maximal, we have $\kappa(H) \leq \overline{\kappa}(H) \leq k$. In order to complete the proof, we only need to show that $\kappa(H) \geq k$.

If $n = k + r - 1$, then H is complete and $\kappa(H) = n - r + 1 = k$. Thus, assume $n \geq k + r$, and so H is not complete. On the contrary, assume $\kappa(H) < k$. Since H is not complete, H has a vertex-cut S with $|S| = \kappa(H) < k$. Let C_1 be a component of $H - S$ and $C_2 = H - (S \cup V(C_1))$. By $|V(C_1) \cup V(C_2)| = n - |S| \geq k + r - (k-1) = r + 1$ we can choose a r-subset $e \subseteq V(C_1) \cup V(C_2)$ such that $e \cap V(C_i) \neq \emptyset$ for $i = 1, 2$. Then $e \in E(H^c)$.

Since H is vertex-k-maximal, we have $\overline{\kappa}(H + e) \geq k + 1$. Hence $H + e$ contains a subhypergraph H' with $\kappa(H') = \overline{\kappa}(H + e) \geq k + 1$. Since $\overline{\kappa}(H) \leq k$, H' cannot be a subhypergraph of H, and so $e \in E(H')$. Since $V(H') \cap V(C_i) \neq \emptyset$ for $i = 1, 2$ it follows that $V(H') \cap S$ is a vertex-cut of $H' - e$.

Since $|V(C_1) \cup V(C_2)| = n - |S| \geq k + r - (k-1) = r + 1 \geq 3$, one of C_i, say C_1, contains at least two vertices. Let $u_1 \in e \cap V(C_1)$. Then $S' = (V(H') \cap S) \cup \{u_1\}$ is a vertex-cut of H', and so we obtain

$$k + 1 > |S| + 1 \geq |V(H') \cap S| + 1 = |S'| \geq \kappa(H') \geq k + 1,$$

a contradiction.

Let H be a vertex-k-maximal r-uniform hypergraph with $|V(H)| \geq k + r$. By Lemma 2.1, $\overline{\kappa}(H) = \kappa(H) = k$. By $|V(H)| \geq k + r$, H is not complete, thus contains vertex-cuts. Let S be a minimum vertex-cut of H, C_1 be a component of $H - S$ and $C_2 = H - (S \cup V(C_1))$. We call (S, H_1, H_2) a *separation triple* of H, where $H_1 = H[S \cup V(C_1)]$ and $H_2 = H[S \cup V(C_2)]$.

Lemma 2.2 *Let n, k, r be integers with $k, r \geq 2$ and $n \geq k + r$, and H be a vertex-k-maximal r-uniform hypergraph on n vertices. Assume (S, H_1, H_2) is a separation triple of H. If $e \in E(H_1^c) \cup E(H_2^c)$, then any subhypergraph H' of $H + e$ with $\kappa(H') \geq k + 1$ is either a subhypergraph of $H_1 + e$ or a subhypergraph of $H_2 + e$. Furthermore, if $e \subseteq E(H_i^c) \setminus E((H[S])^c)$, then H' is a subhypergraph of $H_i + e$ for $i = 1, 2$.*

Proof Let $e \in E(H_1^c) \cup E(H_2^c)$. Since H is vertex-k-maximal, we have $\overline{\kappa}(H + e) \geq k+1$. Let H' be a subhypergraph of $H+e$ with $\kappa(H') = \overline{\kappa}(H+e) \geq k+1$. We assume, on the contrary, that $V(H') \cap (V(H_1) - S) \neq \emptyset$ and $V(H') \cap (V(H_2) - S) \neq \emptyset$. This, together with $e \in E(H_1^c) \cup E(H_2^c)$, implies that $S \cap V(H')$ is a vertex-cut of H'. Hence $k = |S| \geq |S \cap V(H')| \geq \kappa(H') \geq k + 1$, a contradiction. Therefore, we cannot have both $V(H') \cap (V(H_1) - S) \neq \emptyset$ and $V(H') \cap (V(H_2) - S) \neq \emptyset$. If $V(H') \cap (V(H_1) - S) = \emptyset$, then H' is a subhypergraph of $H_2 + e$; if $V(H') \cap (V(H_2) - S) = \emptyset$, then H' is a subhypergraph of $H_1 + e$.

If $e \subseteq E(H_1^c) \setminus E((H[S])^c)$, then $V(H') \cap (V(H_1) - S) \neq \emptyset$ and $V(H') \cap (V(H_2) - S) = \emptyset$, thus H' is a subhypergraph of $H_1 + e$. Similarly, if $e \subseteq E(H_2^c) \setminus E((H[S])^c)$, then H' is a subhypergraph of $H_2 + e$. □

Lemma 2.3 *Let n, k, r be integers with $k, r \geq 2$ and $n \geq k + r$, and H be a vertex-k-maximal r-uniform hypergraph on n vertices. Assume (S, H_1, H_2) is a separation triple of H and $n_i = |V(H_i)|$ for $i = 1, 2$. Then*

(i) $E_{H^c}[V(H_1) - S, S, V(H_2) - S] = \emptyset$, *and*
(ii) $d_H(V(H_1) - S, S, V(H_2) - S) = \binom{n}{r} - \binom{n_1}{r} - \binom{n_2}{r} + \binom{k}{r} - \binom{n-k}{r} + \binom{n_1-k}{r} + \binom{n_2-k}{r}$.

Proof (i) By contradiction, assume $E_{H^c}[V(H_1) - S, S, V(H_2) - S] \neq \emptyset$. Let $e \in E_{H^c}[V(H_1) - S, S, V(H_2) - S]$. Since H is vertex-k-maximal, there is a subhypergraph H' of $H + e$ such that $\kappa(H') = \overline{\kappa}(H + e) \geq k + 1$. By $\overline{\kappa}(H) \leq k$, $e \in E(H')$. This, together with $e \in E_{H^c}[V(H_1) - S, S, V(H_2) - S]$, implies $V(H') \cap S \neq \emptyset$ and $V(H') \cap (V(H_i) - S) \neq \emptyset$ for $i = 1, 2$. Hence $S \cap V(H')$ is a vertex-cut of H'. But then we obtain $k = |S| \geq |S \cap V(H')| \geq \kappa(H') \geq k + 1$, a contradiction. It follows $E_{H^c}[V(H_1) - S, S, V(H_2) - S] = \emptyset$.

(ii) By (i), $E_{H^c}[V(H_1) - S, S, V(H_2) - S] = \emptyset$. This implies that if e is a r-subset such that $e \cap S \neq \emptyset$ and $e \cap (V(H_i) - S) \neq \emptyset$ for $i = 1, 2$, then $e \in E(H)$. Since the number of r-subsets contained in $V(H_1)$ or $V(H_2)$ is $\binom{n_1}{r} + \binom{n_2}{r} - \binom{k}{r}$, and the number of r-subsets exactly intersecting $V(H_1) - S$ and $V(H_1) - S$ is $\binom{n-k}{r} - \binom{n_1-k}{r} - \binom{n_2-k}{r}$, we have

$$
\begin{aligned}
& d_H(V(H_1) - S, S, V(H_2) - S) \\
&= |E_H[V(H_1) - S, S, V(H_2) - S]| \\
&= \binom{n}{r} - \left(\binom{n_1}{r} + \binom{n_2}{r} - \binom{k}{r}\right) - \left(\binom{n-k}{r} - \binom{n_1-k}{r} - \binom{n_2-k}{r}\right) \\
&= \binom{n}{r} - \binom{n_1}{r} - \binom{n_2}{r} + \binom{k}{r} - \binom{n-k}{r} + \binom{n_1-k}{r} + \binom{n_2-k}{r}.
\end{aligned}
$$

This completes the proof.

3 The Lower Bound of the Sizes of Vertex-k-Maximal r-Uniform Hypergraphs

The *union* of two hypergraphs H_1 and H_2, denoted by $H_1 \cup H_2$, is the hyperg with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. The *r-join* of hypergraphs H_1 and H_2, denoted by $H_1 \vee_r H_2$, is the hypergraph obtained fron union of H_1 and H_2 by adding all the edges with cardinality r that connect the ver of H_1 with H_2.

Definition 1 Let n, k, r be integers such that $k, r \geq 2$ and $n \geq k+1$. We d $H_L(n; k, r)$ to be $K_k^r \vee_r (K_{n-k}^r)^c$.

Lemma 3.1 *Let n, k, r be integers such that $k, r \geq 2$ and $n \geq k+1$. If I $H_L(n; k, r)$, then*

(i) *H is vertex-k-maximal, and*
(ii) $|E(H)| = \binom{n}{r} - \binom{n-k}{r}$.

Proof (i) By Definition 1, H is obtained from the union of K_k^r and $(K_{n-k}^r)^c$ by ad all edges with cardinality r connecting $V(K_k^r)$ with $V((K_{n-k}^r)^c)$.
Since $V(K_k^r)$ is a vertex-cut of H and $H - V(K_k^r) = (K_{n-k}^r)^c$, there i subhypergraph with vertex-connectivity at least $k+1$, and so $\overline{\kappa}(H) \leq$ $E(H^c) = \emptyset$, then H is vertex-k-maximal by the definition of vertex-k-max hypergraph. If $E(H^c) \neq \emptyset$, then for any $e \in E(H^c)$, e must be containe $V((K_{n-k}^r)^c)$, and so $(H+e)[V(K_k^r) \cup e]$ is isomorphic to K_{k+r}^r and $\kappa(($ $e)[V(K_k^r) \cup e]) = k+1$. That is $\overline{\kappa}(H+e) \geq k+1$. Thus H is vertex-k-maxi
(ii) holds by a direct calculation.

Theorem 3.2 *Let n, k, r be integers such that $k, r \geq 2$ and $n \geq k+1$. If H is ve k-maximal, then $|E(H)| \geq \binom{n}{r} - \binom{n-k}{r}$.*

Proof We will prove the theorem by induction on n. If $n \leq k+r-1$, the H is vertex-k-maximal, we have $H \cong K_n^r$. Thus $|E(H)| = \binom{n}{r} = \binom{n}{r} - \binom{n-k}{r}$ $n-k \leq r-1$.

Now we assume that $n \geq k+r$, and that the theorem holds for smaller value Since H is vertex-k-maximal and $n \geq k+r$, we have H is not complete. By Lemm $\overline{\kappa}(H) = \kappa(H) = k$, and so H has a separation triple (S, H_1, H_2) with $|S| = k$ $n_1 = |V(H_1)|$ and $n_2 = |V(H_2)|$. Then $n_1, n_2 \geq k+1$ and $n = n_1 + n_2 - k$.

Since H is vertex-k-maximal, for any $e \in E((H[S])^c)$, there is a $(k+1)$-ve connected subhypergraph H' of $H+e$. By Lemma 2.2, H' is either a subhyperg of $H_1 + e$ or a subhypergraph $H_2 + e$. Define

$$E_1 = \{e : e \in E((H[S])^c) \text{ and } \overline{\kappa}(H_1 + e) = k\}$$
$$E_2 = \{e : e \in E((H[S])^c) \text{ and } \overline{\kappa}(H_2 + e) = k\}$$

Claim. *Each of the following holds.*

(i) $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 \subseteq E((H[S])^c)$.
(ii) There is a subset $E_1' \subseteq E_1$ such that $H_1 + E_1'$ is vertex-k-maximal.
(iii) There is a subset $E_2' \subseteq E_2$ such that $H_2 + E_2'$ is vertex-k-maximal.

By the definition, $E_1 \cup E_2 \subseteq E((H[S])^c)$. Since H is vertex-k-maximal, we have $E_1 \cap E_2 = \emptyset$, and so Claim (i) holds.

Assume first that $H_1 + E_1$ is complete. If $n_1 \le k+r-1$, then $\overline{\kappa}(H_1+E_1) \le k$, and so $H_1 + E_1$ is vertex-k-maximal by the definition of vertex-k-maximal hypergraphs. If $n_1 \ge k+r$, then by $\overline{\kappa}(H_1) \le \overline{\kappa}(H) \le k$ and $\overline{\kappa}(H_1 + E_1) \ge k+1$, we can choose a maximum subset $E_1' \subseteq E_1$ such that $\overline{\kappa}(H_1 + E_1') \le k$. It follows by the maximality of E_1' and by the definition of vertex-k-maximal hypergraphs that $H_1 + E_1'$ is vertex-k-maximal. Next, we assume $H_1 + E_1$ is not complete. Take an arbitrary edge $e \in E((H_1+E_1)^c)$. Then $e \in E(H^c)$, and so as H is vertex-k-maximal, $H+e$ contains a $(k+1)$-vertex-connected subhypergraph H' with $e \in E(H')$. If $e \cap (V(H_1)-S) \ne \emptyset$, then by Lemma 2.2, H' is a subhypergraph of H_1+e. If $e \subseteq S$, then as $e \notin E_1$, we can choose H' such that H' is a subhypergraph of H_1+e. That is, $\overline{\kappa}(H_1+E_1+e) \ge k+1$. If $\overline{\kappa}(H_1+E_1) \le k$, then $H_1 + E_1$ is vertex-k-maximal. If $\overline{\kappa}(H_1 + E_1) \ge k+1$, then by $\overline{\kappa}(H_1) \le \overline{\kappa}(H) \le k$, we can choose a maximum subset $E_1' \subseteq E_1$ such that $\overline{\kappa}(H_1 + E_1') \le k$. It also follows by the maximality of E_1' and by the definition of vertex-k-maximal hypergraphs that $H_1 + E_1'$ is vertex-k-maximal. This verifies Claim (ii). By symmetry, Claim (iii) holds. Thus the proof of the Claim is complete.

By Claim (ii) and Claim (iii), there are $E_1' \subseteq E_1$ and $E_2' \subseteq E_2$ such that H_1+E_1' and H_2+E_2' are vertex-k-maximal. Since $n_1, n_2 \ge k+1$, by induction assumption, we have $|E(H_1+E_1')| \ge \binom{n_1}{r} - \binom{n_1-k}{r}$ and $|E(H_2+E_2')| \ge \binom{n_2}{r} - \binom{n_2-k}{r}$. By Claim (i) and the definition of $(H[S])^c$, we have $|E_1'|+|E_2'|+|E(H[S])| \le |E_1|+|E_2|+|E(H[S])| \le |E((H[S])^c)| + |E(H[S])| = \binom{k}{r}$. Thus

$$
\begin{aligned}
|E(H)| &= |E(H_1)| + |E(H_2)| - |E(H[S])| + |E_H[V(H_1) - S, S, V(H_2) - S]| \\
&= |E(H_1 + E_1')| - |E_1'| + |E(H_2 + E_2')| \\
&\quad - |E_2'| - |E(H[S])| + |E_H[V(H_1) - S, S, V(H_2) - S]| \\
&\ge \binom{n_1}{r} - \binom{n_1-k}{r} + \binom{n_2}{r} - \binom{n_2-k}{r} - \binom{k}{r} \\
&\quad + \binom{n}{r} - \binom{n_1}{r} - \binom{n_2}{r} + \binom{k}{r} - \binom{n-k}{r} + \binom{n_1-k}{r} + \binom{n_2-k}{r} \text{ (By Lemma 2.3)} \\
&= \binom{n}{r} - \binom{n-k}{r}.
\end{aligned}
$$

This proves Theorem 3.2. □

By Lemma 3.1, the lower bound of the sizes of vertex-k-maximal hypergraphs given in Theorem 3.2 is best possible. If $r = 2$, then a r-uniform hypergraph H is just a graph. Thus Theorem 1.2 is a corollary of Theorem 3.2.

Corollary 3.3 (Xu, Lai and Tian [17]) *Let n, k be integers with $n \ge k+1 \ge 3$. If G is a vertex-k-maximal graph on n vertices, then $|E(G)| \ge \binom{n}{2} - \binom{n-k}{2} = (n-k)k + \frac{k(k-1)}{2}$. Furthermore, this bound is best possible.*

4 The Upper Bound of the Sizes of Vertex-k-Maximal r-Uniform Hypergraphs

Definition 2 Let n, k, r be integers such that $k, r \geq 2$ and $n \geq 2k$. Assume $n = pk+$ where p, q are integers and $1 \leq q \leq k$. We define $H_U(n; k, r)$ to be $((p-1)K_k^r$ $K_q^r) \vee_r (K_k^r)^c$, where $(p-1)K_k^r$ is the union of $p-1$ complete r-uniform hypergrap on k vertices.

Lemma 4.1 *Let n, k, r be integers such that $k, r \geq 2$ and $n \geq 2k$. If $H = H_U(n; k,$ then*

(i) *H is vertex-k-maximal, and*
(ii) *$|E(H)| \leq \binom{n}{r} - \binom{n-k}{r} + (\frac{n}{k} - 2)\binom{k}{r}$, where the equality holds if n is a multiple k.*

Proof (i) By Definition 2, $H = ((p-1)K_k^r \cup K_q^r) \vee_r (K_k^r)^c$. Denote the $p-1$ co plete r-uniform hypergraphs on k vertices by $K_k^r(1), \ldots, K_k^r(p-1)$. Let H_0 $H[V((K_k^r)^c)]$, $H_p = H[V(K_q^r)]$ and $H_i = H[V(K_k^r(i))]$ for $1 \leq i \leq p-$ Then $H = H_0 \vee_r (H_1 \cup \cdots \cup H_p)$.
Since $V(H_0)$ is a vertex-cut of size k and every component of $H - V(H_0)$ l at most k vertices. It follows that H contains no $(k+1)$-vertex-connected sı hypergraphs, and so $\overline{\kappa}(H) \leq k$. If $E(H^c) = \emptyset$, then H is vertex-k-maximal the definition of vertex-k-maximal hypergraphs. Thus we assume $E(H^c) \neq$ in the following. Let $e \in E(H^c)$. If $e \subseteq V(H_0)$, then $H' = H[V(H_1) \cup e]$ isomorphic to K_{k+r}^r, and so $\kappa(H') = k+1$. If $e \subseteq V(H_1) \cup \cdots \cup V(H_p)$, le be exact-$(V(H_{i1}), \ldots, V(H_{is}))$-crossing. We will prove that $H'' = H[V(H_0$ $V(H_{i1}) \cup \cdots \cup (H_{is})] + e$ is $(k+1)$-vertex-connected. It suffices to prove t $H'' - S$ is connected for any $S \subseteq V(H'')$ with $|S| = k$. If $S = V(H_0)$, th by e is exact-$(V(H_{i1}), \ldots, V(H_{is}))$-crossing, $H'' - S$ is connected. So assu $V_0' = V(H_0) \setminus S \neq \emptyset$. Let $V_1' = (V(H_{i1}) \cup \cdots \cup V(H_{is})) \setminus S$. Then H'' - is isomorphic to $H[V_0'] \vee_r H[V_1']$ if $S \cap e \neq \emptyset$; and $H'' - S$ is isomorphic $H[V_0'] \vee_r H[V_1'] + e$ if $S \cap e = \emptyset$. Since $V_0', V_1' \neq \emptyset$ and $|V_0' \cup V_1'| \geq r$, we obt that $H'' - S$ is connected. Thus $\overline{\kappa}(H+e) \geq k+1$ for any $e \in E(H^c)$, and sc is vertex-k-maximal.

(ii) By a direct calculation, we have $|E(H)| \leq \binom{n}{r} - \binom{n-k}{r} + (\frac{n}{k} - 2)\binom{k}{r}$, where equality holds if n is a multiple of k.

Motivated by Conjecture 1, we propose the following conjecture for verte› maximal r-uniform hypergraphs.

Conjecture 2 *Let k, r be integers with $k, r \geq 2$. Then for sufficiently large n, e› vertex-k-maximal r-uniform hypergraph H on n vertices satisfies $|E(H)| \leq \binom{n}{r}$ $\binom{n-k}{r} + (\frac{n}{k} - 2)\binom{k}{r}$.*

The following theorem confirms Conjecture 2 for the case $k < r$.

Theorem 4.2 *Let n, k, r be integers such that $k, r \geq 2$ and $n \geq 2k$. If $k < r$, ı every vertex-k-maximal r-uniform hypergraph H on n vertices satisfies $|E(H)|$ $\binom{n}{r} - \binom{n-k}{r} + (\frac{n}{k} - 2)\binom{k}{r} = \binom{n}{r} - \binom{n-k}{r}$.*

Proof We will prove the theorem by induction on n. If $n \le k+r-1$, then by H is vertex-k-maximal, we have $H \cong K_n^r$. Thus $|E(H)| = \binom{n}{r} = \binom{n}{r} - \binom{n-k}{r}$ by $n-k \le r-1$.

Now we assume that $n \ge k+r$, and that the theorem holds for smaller value of n. Since H is vertex-k-maximal and $n \ge k+r$, we have H is not complete. Let S be a minimum vertex-cut of H. By Lemma 2.1, $|S| = k$. Let C_1 be a minimum component of $H-S$ and $C_2 = H - (V(C_1) \cup S)$. Assume $H_1 = H[V(C_1) \cup S]$ and $H_2 = H[V(C_2) \cup S]$. Since $k < r$, we have $E((H[S])^c) = \emptyset$, and so H_1 and H_2 are both vertex-k-maximal by Lemma 2.2. Let $n_1 = |V(H_1)|$ and $n_2 = |V(H_2)|$. Then $n = n_1 + n_2 - k$ and $k+1 \le n_1 \le n_2$. We consider two cases in the following.

Case 1. $|V(C_1)| = 1$.

By $|V(C_1)| = 1$, we obtain that $n_2 = n-1 \ge k+r-1 \ge 2k$. Since H_2 is vertex-k-maximal, by induction assumption, we have $|E(H_2)| \le \binom{n-1}{r} - \binom{n-k-1}{r}$. Thus

$$\begin{aligned}|E(H)| &= |E(H_1)| + |E(H_2)| - |E(H[S])| + |E_H[V(H_1) - S, S, V(H_2) - S]| \\ &\le \tbinom{k}{r-1} + \tbinom{n-1}{r} - \tbinom{n-k-1}{r} + \tbinom{n-1}{r-1} - \tbinom{k}{r-1} - \tbinom{n-k-1}{r-1} \\ &= \tbinom{n}{r} - \tbinom{n-k}{r}.\end{aligned}$$

Case 2. $|V(C_1)| \ge 2$.

By $|V(C_1)| \ge 2$, we obtain that C_1 contains edges, and so $|V(C_1)| \ge r$. Thus $n_2 \ge n_1 \ge k+r \ge 2k+1$. Since both H_1 and H_2 are vertex-k-maximal, by induction assumption, we have $|E(H_i)| \le \binom{n_i}{r} - \binom{n_i-k}{r}$ for $i = 1, 2$. Thus

$$\begin{aligned}|E(H)| &= |E(H_1)| + |E(H_2)| - |E(H[S])| + |E_H[V(H_1) - S, S, V(H_2) - S]| \\ &\le \tbinom{n_1}{r} - \tbinom{n_1-k}{r} + \tbinom{n_2}{r} - \tbinom{n_2-k}{r} \\ &\quad + \tbinom{n}{r} - \tbinom{n_1}{r} - \tbinom{n_2}{r} + \tbinom{k}{r} - \tbinom{n-k}{r} + \tbinom{n_1-k}{r} + \tbinom{n_2-k}{r} \text{ (By Lemma 2.3)} \\ &= \tbinom{n}{r} - \tbinom{n-k}{r}.\end{aligned}$$

This completes the proof. □

Combining Theorem 3.2 with Theorem 4.2, we have the following corollary.

Corollary 4.3 *Let n, k, r be integers such that $k, r \ge 2$ and $n \ge 2k$. If $k < r$, then every vertex-k-maximal r-uniform hypergraph H on n vertices satisfies $|E(H)| = \binom{n}{r} - \binom{n-k}{r}$.*

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