Homework 2. Due day: 9/15/06

(2.1) Let $G$ be a graph with $E(G) \neq \emptyset$. A subset $X \subseteq E(G)$ is a bicircuit if the edge induced subgraph $X$ is connected with $|X| = |V(X)| + 1$ and with minimum degree $\delta(X) \geq 2$. A connected component $L$ of $G$ is an acyclic component of $G$ if $|E(L)| = |V(L)| - 1$. Let $\omega(G)$ denote the number of acyclic components of $G$.

(i) Let $C_B(G)$ denote the collection of all bicircuits of $G$. Show that $C_B(G)$ satisfies the circuit axioms and so there is a matroid $B(G)$ on $E(G)$ with $C(B(G)) = C_B(G)$, called the bicircular matroid of $G$.

(ii) Let $r$ denote the rank function of $B(G)$. Show that $\forall X \subseteq E(G)$, $r(X) = |V(X)| - \omega_n(X)$.

Proof: (i) (C1). Since when $X = \emptyset$, it is not true that $|X| = |V(X)| + 1$, $\emptyset \notin C_B(G)$.

To show (ii) and (iii), we need two lemmas.

Lemma 1 If $G$ is a graph such that $|E(G)| \geq |V(G)| + 1$, then $G$ must contain a bicircuit.

Proof: By contradiction, we assume that $G$ is a countexample with $|V(G)|$ minimized. We assume that $G$ is connected, otherwise we argue componentwise. Since $G$ is connected, $G$ has a spanning tree $T$. Note that $|E(G)| - 1 \geq |V(G)| = |V(T)| = |E(T)| + 1$. There exists $e, e' \in E(G) - E(T)$. Let $H = T + \{e, e'\}$. Then $|E(H)| = |V(H)| + 1$. By the minimality of $G$, we must have $G = H$. If $\delta(G) \geq 2$, then $G$ itself is a bicircuit. If $G$ has a vertex $v$ of degree 1, then by the minimality of $G$, $G - v$ has a bicircuit, and so $G$ must contains a bicircuit, contrary to the assumption on $G$.

Lemma 2 Any bicircuit must be either a $\theta$-graph, or a Figure 8 graph, or a dump-bell graph.

Proof: Let $L$ be a bicircuit. Then $L$ contains a spanning tree $T$ with exactly two edges $e, e'$ not in $T$, and so $L$ has two circuits $C_1$ and $C_2$ as the fundamental circuits of $e$ and $e'$ with respect to $T$, respectively. If $C_1 \cap C_2$ contains an edge, (or if $C_1 \cap C_2$ contains exactly one vertex) then $L' = C_1 \cup C_2$ is a $\theta$-graph (or a Figure 8 graph), with $|L'| = |V(L')| + 1$. As $|L| = |V(L)| + 1$, and as $L$ is connected, for any vertex $v$ in $L - L'$, $v$ has exactly one path in $L$ to $L'$, and $\delta(L) = 1$, contrary to the definition of a bicircuit. Thus in this case $L$ is a $\theta$-graph or a Figure 8 graph. If $C_1$ and $C_2$ are vertex disjoint, then $|L'| = |V(L')|$, and as $L$ is connected, there must be a path $P$ in $L$ joining $C_1$ to $C_2$. Let $L'' = C_1 \cup C_2 \cup P$, which is a dump-bell graph with $|L''| = |V(L'')| + 1$. With a similar argument as above, since $L$ is connected with $\delta(L) \geq 2$, and since $|L| = |V(L)| + 1$, we conclude that $L = L''$ in this case, and so it is a dump-bell graph.

(C2). Let $L_1, L_2 \in C_B(G)$ such that $L_1 \neq L_2$ and $L_1 \subset L_2$. Since $L_1$ has two circuits (2-regular connected nontrivial graphs) $C_1$ and $C_2$, $C_1 \cap C_2$ may or may not be empty). Since $L_1 \subseteq L_2$, then $C_1$ and $C_2$ are also subgraphs of $L_2$. Let $e \in L_2 - L_1$, we must have $e \in (C_1 \cup C_2)$, and so the only possibility is when $L_2$ is the dump-bell shape bicircuit, and $e$ lies in the path $P$ joining the two circuits of the dumpbell. Since $L_1 \subseteq L_2$, $C_1$ and $C_2$ are also vertex disjoint circuits in $L_1$, and so as $L_1$ must be connected and as $L_1 \subseteq L_2$, $P$ must also be a subgraph of $L_1$, and so $e \in P \subseteq L_1$, contrary to the fact that $e \in L_2 - L_1$.

(C3) Let $L_1, L_2 \in C_B(G)$ with $L_1 \neq L_2$ but $e \in L_1 \cap L_2$. Let $C_1, C_2$ denote the two circuits of $L_1$ and $C_1', C_2'$ denote the two circuits of $L_2$.

If $e \in C_1' \cap C_2'$, then $L' = C_1' \cup C_2' - e$ is a connected graph which contains at least one circuit (by the fact that circuits of a graph satisfy (C3)), and so $|L'| \geq |V(L')|$. Since $L_1$ is connected, $H = L' \cup (L_1 - C_1')$ is also connected with $|H| \geq |V(H)| + 1$ (as at least one vertex is counted in both $|V(L')|$ and in $|V(L_1 - C_1')|$), and so by Lemma 1, $H$ contains a bicircuit $L_3$. Note that $H \subseteq (C_1' \cup C_2' - e) \cup (L_1 - C_1') \subseteq L_1 \cup L_2 - e$, (C3) is proved in this case.

The proofs for the other cases (by symmetry, these are either $e \in C_1' \cap (L_2 - (C_1' \cup C_2'))$ or $e \in (L_1 - (C_1' \cup C_2')) \cap (L_2 - (C_1' \cup C_2'))$) are similar.
Lemma 2, any acyclic components cannot contain a bicircuit, and so

(ii) Let \( X_1, X_2, \ldots, X_n \) be acyclic components of \( X \) and \( X_{n+1}, \ldots, X_m \) are cyclic components of \( X \). By Lemma 2, any acyclic components cannot contain a bicircuit, and so \( r(X_i) = |X_i| = |V(X_i)| - 1 \). Again by Lemma 2, if \( X_j \) is a cyclic component, then \( X_j \) contains a spanning tree plus an edge, which, by Lemma 1, is a maximally independent set in \( B(G) \). Thus in this case, \( r(X_j) = |V(X_j)| \). Summing up along the components,

\[
r(X) = \sum_{i=1}^{m} r(X_i) = \sum_{i=1}^{m} |V(X_i)| - a = |V(X)| - \omega_d(X).
\]

(2.2) Let \( M \) be a matroid with rank function \( r \) and closure operator \( cl \). \( \forall X, Y \subseteq E(M) \), each of the following holds.

(i) If \( X \subseteq cl(Y) \subseteq cl(X) \), then \( cl(X) = cl(Y) \).

(ii) If \( cl(X) = cl(Y) \), then \( r(X) = r(Y) \).

(iii) \( r(X \cup Y) = r(X \cup cl(Y)) = r(cl(X \cup Y)) = r(cl(X) \cup Y) \).

Proof: (i) By (CL2) and since \( X \subseteq cl(Y) \subseteq cl(X) \), we have \( cl(X) \subseteq cl(cl(Y)) \subseteq cl(X) \). By (CL3), both \( cl(cl(X)) = cl(X) \) and \( cl(cl(Y)) = cl(Y) \); and so \( cl(X) \subseteq cl(Y) \subseteq cl(X) \). Thus equality must hold.

(ii) Let \( B_X \in B(M[X]) \). Then \( r(X) = |B_X| = r(B_X) \). Note that \( B_X \subseteq X \subseteq cl(X) \). We can augment \( B_X \) to a basis \( B \in B(M) \). Note that \( r(W) = r(cl(W)) \), \( \forall W \subseteq E \) (shown in class), \( r(X) = |B_X| \leq |B| = r(cl(Y)) = r(cl(X)) = r(X) \), and so \( r(X) = r(cl(Y)) = r(Y) \).

(iii) Note that for any \( W \subseteq E \), \( r(W) = r(cl(W)) \), and so we already have \( r(X \cup Y) = r(cl(X \cup Y)) \).

Let \( B_Y \in B(M[Y]) \). Then \( |B_Y| = r(Y) = r(cl(Y)) \), and so \( B_Y \in B(M) \). Augment \( B_Y \) to \( B' \in B(M[X \cup cl(Y)]) \). Then \( B' - B_Y \subseteq X - cl(Y) \), and so \( B' \in I(M[X \cup Y]) \). Thus \( r(X \cup Y) \leq r(X \cup cl(Y)) = |B'| \leq r(X \cup Y) \), and so \( r(X \cup Y) = r(X \cup cl(Y)) \).

Replacing \( X \) by \( cl(Y) \), and \( Y \) by \( X \) in \( r(X \cup Y) = r(X \cup cl(Y)) \), we obtain \( r(X \cup cl(Y)) = r(cl(X) \cup cl(Y)) \).

(2.3) Let \( M \) be a matroid, let \( B \in B(M) \) and \( B^* = E - B \).

(i) If \( e \in B \), then \( C^* = C_M(e, B^*) \) is the only circuit in \( C(M^*) \) satisfying \( C^* \cap (B - e) = \emptyset \).

(ii) Suppose that \( f \in B^* \) and \( e \in B \). Show that \( f \in C_M(e, B^*) \) if and only if \( e \in C_M(f, B) \).

Proof: (i) Let \( C^* = C^*(e, B^*) \) and assume that \( \exists C^*_1 \in C(M^*) \) such that \( C^*_1 \subseteq E - (B - e) = (E - B) \cup e \). Then \( e \in C^*_1 \) since \( E - B \in B(M^*) \). Therefore, \( e \in C^* \cap C^*_1 \). By (C3), \( \exists C^*_2 \in C(M^*) \) such that \( C^*_2 \subseteq (C^*_1 \cup C^*) \cap C^* \). \( C^*_2 \subseteq (C^*_1 \cup C^*) \cap (E - B), \) contrary to \( E - B \in B(M^*) \).

(ii) Suppose that \( f \in C_M(e, B^*) \). Note that \( B = E - B^* \). Then \( (B^* \cup e - f) \in B(M^*) \) iff \( (B \cup f) - e = (E - (B^* \cup e)) \) iff \( f \in B(M) \) iff \( e \in C_M(f, B) \).

(2.4) Let \( M = M_3[I_4|D] \), where

\[
D = \begin{bmatrix}
0 & 1 & 1 & -1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
-1 & 1 & 1 & 0
\end{bmatrix}.
\]

(i) Show that \( M \cong M^* \).

(ii) Is it true that \( M = M^{**} \)?

Proof: (i) Note that \( D^T = D \). Therefore, \( M^* = M_3[-D^T|I_4] = M_3[D|I_4] \) by multiplying each of the first 4 columns by \((-1)\). Thus column permutations gives \( M^* = M_3[D|I_4] \cong M_3[I_4|D] = M \), and so \( M \) is self-dual.

Let \( X = \{e_2, e_3, e_4, e_5\} \) denote the corresponding 4 columns of \([I_4|D]\). Then computing the determinants we see that \( X \) is dependent in \( M \). However, computing the corresponding determinant in \([D|I_4]\),
we conclude that $X$ is independent in $M^*$, and so $M \neq M^*$, or $M$ is not identically self-dual.

(2.5) Show that $AG(3, 2)$ is a self-dual matroid.

**Proof:** We can show that $AG(3, 2)$ is identically self-dual. It suffices to show every basis of $AG(3, 2)$ is also a cobasis and vice versa. Given a $B^* \in B(M^*)$, $B = E - B^* \in B(M)$, and so one can have a standard representation of $M$ so that $B$ corresponds to an identity matrix. Therefore, it suffices to show that the columns other than this identity matrix are linearly independent. Given a standard representation of $AG(3, 2)$:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1
\end{bmatrix}, \quad \text{and then} \quad 
\begin{vmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1
\end{vmatrix} = -1 \neq 0.