Dicycle Cover of Hamiltonian Oriented Graphs

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1. The Problem

We consider finite loopless graphs and digraphs, and unde-

fined notations and terms will follow [1] for graphs and [2] for digraphs. In particular, a cycle is a 2-regular con-

nected nontrivial graph. A cycle cover of a graph is a collection of cycles of a graph such that \( E(G) = \bigcup_{C \in \mathcal{C}} E(C) \). Bondy [3] conjectured that if \( G \) is a 2-connected simple graph with \( n \geq 3 \) vertices, then \( G \) has a cycle cover \( \mathcal{C} \) with \( |\mathcal{C}| \leq (2n - 3)/3 \). Bondy [3] showed that this conjecture, if proved, would be best possible. Luo and Chen [4] proved that this conjecture holds for 2-connected simple cubic graphs. It has been shown that, for plane triangulations, serial-parallel graphs, or planar graphs in general, one can have a better bound for the number of cycles used in a cover [5–8]. Barnette [9] proved that if \( G \) is a 3-connected simple planar graph of order \( n \), then the edges of \( G \) can be covered by at most \((n+1)/2\) cycles. Fan [10] settled this conjecture by showing that it holds for all simple 2-connected graphs. The best possible number of cycles needed to cover cubic graphs has been obtained in [11, 12].

A directed path in a digraph \( D \) from a vertex \( u \) to a vertex \( v \) is called a \((u, v)\)-dipath. To emphasize the distinction between graphs and digraphs, a directed cycle or path in a digraph is often referred to as a dipath. It is natural to consider the number of dicycles needed to cover a digraph. Following [2], for a digraph \( D, V(D) \) and \( A(D) \) denote the vertex set and arc set of \( D \), respectively. If \( A' \subseteq A(D) \), then \( D[A'] \) is the subdigraph induced by \( A' \). Let \( K^*_n \) denote the complete digraph on \( n \) vertices. Any simple digraph \( D \) on \( n \) vertices can be viewed as a subdigraph of \( K^*_n \). If \( W \) is an arc subset of \( A(K^*_n) \), then \( D + W \) denotes the digraph \( K^*_n[A(D) \cup W] \).

A digraph \( D \) is strong if, for any distinct \( u, v \in V(D), D \) has a \((u, v)\)-dipath. As in [2], \( A(D) \) denotes the arc-strong-connectivity of \( D \). Thus a digraph \( D \) is strong if and only if \( \lambda(D) \geq 1 \). We use \((u, v)\) denoting an arc with tail \( u \) and head \( v \). For \( X, Y \subseteq V(D) \), we define

\[
(X, Y)_D = \{(x, y) \in A(D) : x \in X, \ y \in Y\};
\]

\[
\mathcal{D}_D^-(X) = (X, V(D) - X)_D.
\]

Let

\[
d_D^-(X) = |\mathcal{D}_D^-(X)|,
\]

\[
d_D^+(X) = |\mathcal{D}_D^+(X)|.
\]

When \( X = \{v\} \), we write \( d_D^-(v) = |\mathcal{D}_D^-(v)| \) and \( d_D^+(v) = |\mathcal{D}_D^+(v)| \). Let \( N_D^+(v) = \{u \in V(D) - v : (v, u) \in A(D)\} \) and \( N_D^-(v) = \{u \in V(D) - v : (u, v) \in A(D)\} \) denote the out-neighbourhood and in-neighbourhood of \( v \) in \( D \), respectively. We call the vertices in \( N_D^+(v) \) and \( N_D^-(v) \) the out-neighbours and the in-neighbours of \( v \). Thus, for a digraph \( D \), \( \lambda(D) \geq 1 \) if and only if, for any proper nonempty subset \( O \neq X \subset V(D), |\mathcal{D}_D^-(X)| \geq 1 \).
A dicycle cover of a digraph $D$ is a collection $\mathcal{C}$ of dicycles of $D$ such that $\bigcup_{\mathcal{C} \in \mathcal{C}} A(C) = A(D)$. If $D$ is obtained from a simple undirected graph $G$ by assigning an orientation to the edges of $G$, then $D$ is an oriented graph. The main purpose is to investigate the number of dicycles needed to cover a Hamiltonian oriented graph. We prove the following.

**Theorem 1.** Let $D$ be an oriented graph on $n$ vertices and $m$ arcs. If $D$ has a Hamiltonian dicycle, then $D$ has a dicycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq m - n + 1$. This bound is best possible.

In the next section, we will first show that every Hamiltonian oriented graph with $n$ vertices and $m$ arcs can be covered by at most $m - n + 1$ dicycles. Then we show that, for every Hamiltonian graph with $n$ vertices and $m$ edges, there exists an orientation $D = D(G)$ of $G$ such that any dicycle cover of $D$ must have at least $m - n + 1$ dicycles.

**2. Proof of the Main Result**

In this section, all graphs are assumed to be simple. We start with an observation, stated as lemma below. A digraph $D$ is weakly connected if the underlying graph of $D$ is connected.

**Lemma 2.** A weakly connected digraph $D$ has a dicycle cover if and only if $\lambda(D) \geq 1$.

**Proof.** Suppose that $D$ has a dicycle cover $\mathcal{C}$. If $D$ is not strong, then there exists a proper nonempty subset $\emptyset \neq X \subset V(D)$ such that $|\partial_D^+(X)| = 0$. Since $D$ is weakly connected, $D$ contains an arc $(u, v) \in (V(D) - X, X)_D$. Since $\mathcal{C}$ is a dicycle cover of $D$, there exists a dicycle $C \in \mathcal{C}$ with $(u, v) \in A(C)$. Since $(u, v) \notin (V(D) - X, X)_D$, we conclude that $\emptyset \neq A(C) \cap (X, (V(D) - X))_D \subseteq \partial_D^*(X)$, contrary to the assumption that $|\partial_D^+(X)| = 0$. This proves that $D$ must be strong.

Conversely, assume that $D$ is strong. For any arc $a = (u, v) \in A(D)$, since $D$ is strong, there must be a directed $(v, u)$-path $P$ in $D$. It follows that $C_a = P + a$ is a dicycle of $D$ containing $a$, and so $|C_a : a \in A(D)|$ is a dicycle cover of $D$.

Let $C$ be a dicycle and let $a = (u, v)$ be an arc not in $A(C)$ but with $u, v \in V(C)$. Then $C_a + a$ contains a unique dicycle $C_a$ containing $a$. In the following, we call $C_a$ the fundamental dicycle of $a$ with respect to $C$.

**Lemma 3.** Let $D$ be an oriented graph on $n$ vertices and $m$ arcs. If $D$ has a Hamiltonian dicycle, then $D$ has a dicycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq m - n + 1$.

**Proof.** Let $C_0$ denote the directed Hamiltonian cycle of $D$. For each $a \in A(D) - A(C)$, let $C_a$ denote the fundamental dicycle of $a$ with respect to $C$. Then $\mathcal{C} = \{C_0, C_a : a \in A(D) - A(C)\}$ is a dicycle cover of $D$ with $|\mathcal{C}| \leq m - n + 1$.

To prove that Theorem 1 is best possible, we need to construct, for each integer $n \geq 4$, a Hamiltonian oriented graph on $n$ vertices and $m$ arcs $D$ such that any dicycle cover $\mathcal{C}$ of $D$ must have at least $m - n + 1$ dicycles in $\mathcal{C}$.

Let $G$ be a Hamiltonian simple graph. We present a construction of such an orientation $D = D(G)$. Since $G$ is Hamiltonian, we may assume that $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $C = v_1v_2, \ldots, v_nv_1$ is a Hamiltonian cycle of $G$.

**Definition 4.** One defines an orientation $D = D(G)$ as follows.

(i) Orient the edges in the Hamiltonian cycle $C = v_1v_2, \ldots, v_nv_1$ as follows:

\[
(v_{i+1}, v_i) \in A(D), \quad i = 1, 2, \ldots, n-1, \quad (v_1, v_n) \in A(D).
\]

(ii) For each $i = 2, 3, \ldots, n - 2$, and for each $j = i + 2, i + 3, \ldots, n$, assign directions to edges of $G$ not in $E(C)$ as follows:

\[
(v_i, v_j) \in A(D),
\]

if $v_iv_j \in E(G) - E(C), \quad i + 1 < j \leq n,$

\[
(v_1, v_j) \in A(D),
\]

if $v_1v_j \in E(G) - E(C), \quad i + 1 < j \leq n - 1.$

We make the following observations stated in the lemma below.

**Lemma 5.** Each of the following holds for the digraph $D$:

(i) The dicycle $C_0 = v_1v_nv_{n-1}, \ldots, v_3v_2v_1$ is a Hamiltonian dicycle of $D$.

(ii) The digraph $D - A(C_0)$ is acyclic.

(iii) $N_D^+(v_j) = \{v_{j+1}, v_{j-1}\}$, $N_D^-(v_j) = \{v_{j+1}, v_{j-1}\}$, $N_D^-(v_1) = \{v_2\}$.

(iv) The dicycle $C_0$ is the only dicycle of $D$ containing the arc $(v_1, v_n)$.

(v) The dicycle $C_0$ is the unique Hamiltonian dicycle of $D$.

(vi) If $C''$ is a dicycle of $D$, then $C''$ contains at most one arc in $A(D) - A(C_0)$.

**Proof.** (i) follows immediately from Definition 4(i).

(ii) By Definition 4, the labels of the vertices $V(D) = \{v_1, v_2, \ldots, v_n\}$ satisfy $(v_i, v_j) \in A(D) - A(C_0)$ only if $i < j$. It follows (e.g., Section 2.1 of [2]) that $D - A(C_0)$ is acyclic, and so (ii) holds.

(iii) (iii) follows immediately from Definition 4.

(iv) Let $C'$ be a dicycle of $D$ with $(v_1, v_n) \notin A(C')$. Since $(v_1, v_n) \notin A(C') \cap A(C_0)$, we choose the largest label $i \leq n$, such that $(v_1, v_i), (v_0, v_{i+1}), \ldots, (v_{i+1}, v_i) \in A(C') \cap A(C_0)$. Since $C' \neq C_0$, we have $i \geq 3$. Since $C'$ is a dicycle, there must be a vertex $v_j \in V(D)$ such that $(v_j, v_i) \in A(C')$. By the choice of $i$, we must have $(v_j, v_i) \notin A(C_0)$, and so $(v_j, v_i) \in A(D) - A(C_0)$. By Definition 4(iii), we have $i + 2 \leq j \leq n$, contrary to the fact that $C'$ is a dicycle of $D$ containing $(v_1, v_n)$. This proves (iv).

(v) Let $C'$ be a Hamiltonian dicycle of $D$. Since $V(C') = V(D)$, we have $v_n \in V(C')$. We claim that
If \((v_1, v_n) \notin A(C')\), then there exists \(v_i \in V(C)\) (\(i \in \{v_2, v_3, \ldots, v_{n-1}\}\) such that \((v_i, v_n) \notin A(C')\). Hence, \((v_i, v_n), (v_{n-1}, v_{n-2}), \ldots, (v_2, v_1) \in A(C')\). By Definition 4(i) and (ii), \(N^+(v_1) \subseteq \{v_{n-1}, v_{n-3}, \ldots, v_1\}\), contrary to the fact that \(C'\) is a Hamiltonian dicycle of \(D\). Thus, \((v_1, v_n) \in A(C')\).

It follows from Lemma 5(iv) that we must have \(C = C_0\).

By contradiction, we assume that \(D\) has a dicycle \(C''\) which contains two arcs: \(a_1, a_2 \in A(D) - A(C_0)\). Since \(V(D) = \{v_1, v_2, \ldots, v_n\}\), we have that \(a_1 = (v_1, v_i)\) and \(a_2 = (v_j, v_n)\). Without loss of generality and by Lemma 2, we further assume that \(1 \leq i < j < n\).

Let \(t \geq 1\) be the smallest integer such that \(v_t \in V(C'')\). Since \(C''\) is a dicycle of \(D\), there must be \(v_t \in V(C'')\) such that \((v_t, v_j) \in A(C'')\).

By Definition 4, \((v_t, v_j) \in A(C_0)\) and \(s = t + 1 = j < t\). By the choice of \(t\), \(v_t\) can only have \(s \leq t + 1\) and \((v_{t+1}, v_j) \in A(C'')\). Choose the largest integer \(h\) with \(t + 1 \leq h \leq j\) such that \((v_{t+1}, v_j), (v_{t+2}, v_j), \ldots, (v_{h-1}, v_j) \in A(C'')\) \(\cap A(C_0)\). Since \(C''\) is a dicycle, there must be \(v_k \in V(C'')\) with \(1 \leq k \leq n\) such that \((v_k, v_j) \in A(C'')\). By the maximality of \(h\) and by Definition 4(i), \(v_k = v_t\). By Definition 4(ii), \(1 \leq k \leq h - 2\). By the maximality of \(t\), we must have \(t \leq k \leq h - 2\). It follows by \(j > h\) that \(C''\) cannot contain \(a_1 = (v_1, v_i)\), contrary to the assumption. This contradiction justifies (vi).

To complete the proof of Theorem 1, we present the next lemma.

**Lemma 6.** Let \(G\) be a Hamiltonian simple graph. There exists an orientation \(D = D(G)\) such that every dicycle cover of \(D\) must have at least \(m - n + 1\) dicycles.

**Proof.** Let \(G\) be a Hamiltonian graph and let \(D = D(G)\) be the orientation of \(G\) given in Definition 4. For notational convenience, we adopt the notations in Definition 4 and denote \(V(D) = \{v_1, v_2, \ldots, v_n\}\). Thus, by Lemma 5(v), \(C_0 = v_1 v_2 v_3 \ldots v_n v_1\) is the unique Hamiltonian dicycle of \(D\).

Let \(\mathcal{C}\) be a dicycle cover of \(D\). By Lemma 5(iv), we must have \(C_0 \in \mathcal{C}\). For each arc \(a \in A(D) - A(C_0)\), \(\mathcal{C}\) is a dicycle cover of \(D\) such that \(a \in A(C(a))\). By Lemma 5(vi), \(A(C(a)) \cap A(D) - A(C_0) = \{a\}\). It follows that if \(a \neq a' \in A(D) - A(C_0)\), then \(a \neq a' \) implies \(C(a) \neq C(a')\) \(\in \mathcal{C}\). Thus we have \(|\mathcal{C}| \geq |A(D) - A(C_0)| \geq m - n + 1\).

This proves the lemma.

By Lemmas 3 and 6, Theorem 1 follows. We are about to show that Theorem 1 can be applied to obtain dicycle cover bounds for certain families of oriented graphs. Let \(T_n\) denote a tournament of order \(n\). Then \(T_n\) is an oriented graph. Camion [13, 14] proved that every strong tournament is Hamiltonian. Hence the corollary below follows from Theorem 1.

**Corollary 7.** Every strong tournament on \(n\) vertices has a dicycle cover \(\mathcal{C}\) with \(|\mathcal{C}| \leq n(n - 1)/2 - n + 1\). This bound is best possible.

A bipartite graph \(G(A, B)\) with vertex partition \((A, B)\) is balanced if \(|A| = |B|\). If bipartite graph \(G(A, B)\) has a Hamiltonian cycle, then \(G\) is balanced. Let \(K_{n,n}\) be a complete bipartite graph with vertex partition \((A, B)\) and \(|A| = m, |B| = n\); then \(K_{n,n}\) has Hamiltonian cycle if only if \(m = n \geq 2\); that is, \(K_{n,n}\) is balanced. Let \(K_{n,n}\) denote a balanced complete bipartite graph.

**Corollary 8.** Every Hamiltonian orientation of balanced complete bipartite graph \(K_{n,n}\) has a dicycle cover \(C\) with \(|\mathcal{C}| \leq (n - 1)^2\). This bound is best possible.

**Proof.** Since an oriented balanced complete bipartite graph \(K_{n,n}\) has \(n^2\) arcs, so, by Theorem 1, we have \(|\mathcal{C}| \leq n^2 - 2n + 1 = (n - 1)^2\).

To prove the bound is best possible, we need to construct, for each integer \(n \geq 2\), a Hamiltonian oriented balanced complete bipartite graph on \(2n\) vertices such that any dicycle cover \(\mathcal{C}\) of \(K_{n,n}\) must have at least \((n - 1)^2\) dicycles in \(\mathcal{C}\). We may assume that \(V(K_{n,n}) = \cup_{u_i \in V_1} v_{u_i} u_i \), \(v_1, v_2, v_3, \ldots, v_n\) and \(C = u_1 v_1 u_2 v_2, \ldots, u_n v_n u_1\) is a Hamiltonian cycle of \(K_{n,n}\). We construct an orientation \(D_{n,n} = D(K_{n,n})\) as the orientation of Definition 4; thus, by Lemmas 5 and 6, every dicycle cover \(\mathcal{C}\) of \(D_{n,n}\) must have at least \((n - 1)^2\) dicycles. This proves the corollary.

3. Dicycle Covers of 2 Sums of Digraphs

In this section, we will show that Theorem 1 can also be applied to certain non-Hamiltonian digraphs which can be built via 2 sums. We start with 2 sums of digraphs.

**Definition 9.** Let \(D_n = (V(D_n), A(D_n))\) and \(D_m = (V(D_m), A(D_m))\) be two disjoint digraphs; \(a_1 = (v_{i_1}, v_{j_1}) \in A(D_m)\) and \(a_2 = (v_{i_2}, v_{j_2}) \in A(D_n)\). The 2-sums \(D_n \oplus D_m\) of \(D_n\) and \(D_m\) is obtained from the union of \(D_n\) and \(D_m\) by identifying the arcs \(a_1, a_2\); that is, \(v_{i_1} = v_{i_2}\) and \(v_{j_1} = v_{j_2}\).

**Definition 10.** Let \(D_n, D_{n_1}, \ldots, D_{n_s}\) be \(s\) disjoint digraphs with \(n_1, n_2, \ldots, n_s\) vertices, respectively. Let \(D_n \oplus D_{n_1} \oplus \cdots \oplus D_{n_s}\) denote a sequence of \(2\) sums of \(D_n, D_{n_1}, \ldots, D_{n_s}\), that is, \(((D_n \oplus D_{n_1}) \oplus D_{n_2}) \oplus \cdots \oplus D_{n_s} \). Then \(D\) has a dicycle cover \(\mathcal{C}\) with \(|\mathcal{C}| \leq |A(D)| - |V(D)| + 1\). This bound is best possible.

**Proof.** By Theorem 1, \(D_{n_i} (i = 1, 2, \ldots, s)\) has a dicycle cover \(\mathcal{C}_i\) with \(|\mathcal{C}_i| \leq n_i - n_i + 1\). Let \(\mathcal{C} = \bigcup_{i=1}^s \mathcal{C}_i\); then \(|\mathcal{C}| \leq (n_1 - n_1 + 1) + (n_2 - n_2 + 1) + \cdots + (n_s - n_s + 1) = (n_1 + n_2 + \cdots + n_s) - (n_1 + n_2 + \cdots + n_s) + 1 = |A(D)| - |V(D)| + 1. By Definition 10, \(\mathcal{C}\) is a dicycle cover of \(D\). Thus, \(D\) has a dicycle cover \(\mathcal{C}\) with \(|\mathcal{C}| \leq |A(D)| - |V(D)| + 1\).
Let $G_n$ be $s$ disjoint Hamiltonian simple graphs for $i \in \{1, 2, \ldots, s\}$. We may assume that $V(G_n) = \{v_{i1}, v_{i2}, \ldots, v_{in}\}$ and $C_i = v_{i1}v_{i2}, \ldots, v_{in}v_{i1}$ is a Hamiltonian cycle of $G_n$, and let

$$D_n = D(G_n)$$

be the orientation of $G_n$, given in Definition 4.

For notational convenience, we adopt the notations in Definition 4 and denote $V(D_n) = \{v_{11}, v_{12}, \ldots, v_{sn}\}$. Thus, by Lemma 5(v), $C_{in} = v_{i1}v_{in}, \ldots, v_{in}v_{i1}$ is the unique Hamiltonian dicycle of $D_n$. Let $a_i = (v_{ij}, v_{ik})$ be an arc of $D_n$. We construct the 2-sum digraph $D_n \oplus D_n \oplus \cdots \oplus D_n$ from the union of $D_n$, $D_n$, $D_n$, by identifying the arcs $a_1, a_2, \ldots, a_s$ such that $v_{ij} = v_{ik} = \cdots = v_{ij}$ and $v_{ij} = v_{ik} = \cdots = v_{ij}$. We assume that $v_{ij} = v_{ij} = v_{ij} = \cdots = v_{ij}$ and $v_{ij} = v_{ij} = v_{ij} = \cdots = v_{ij}$ (the case when $s = 2$ is depicted in Figure 1).

Claim I. There does not exist a dicycle whose arcs intersect arcs in two or more $D_n$'s ($i = 1, 2, \ldots, s$).

By Definition 9, we have $V(D_n) \cap V(D_n) = \{v_i, v_j\}$ ($i \neq j$). Without loss of generality, we consider oriented graphs $D_n$ and $D_n$; suppose that there exists a dicycle $C_0$ such that

$$[A(C_0) - (v_2, v_1)] \cap A(D_n) \neq \emptyset,$$

$$[A(C_0) - (v_2, v_1)] \cap A(D_n) \neq \emptyset.$$  

Thus, there must exist four different arcs

$$\{(v_{2}, v_1), (v_2, v_1), (v_2, v_2), (v_2, v_1)\} \in A(C_0)$$

with $(v_{2}, v_1), (v_2, v_2), (v_2, v_1) \in A(D_n)$ and $(v_2, v_2), (v_2, v_1) \in A(D_n)$, as shown in Figure 2, or four different arcs

$$\{(v_{2}, v_1), (v_2, v_2), (v_2, v_2), (v_2, v_2)\} \in A(C_0)$$

with $(v_{2}, v_1), (v_2, v_2), (v_2, v_2) \in A(D_n)$ and $(v_2, v_2), (v_2, v_2) \in A(D_n)$, as shown in Figure 3.

By Definition 9, Lemma 5(iii), and (6), we have $N_j(v_i) = \{v_j\}$, and so $v_{ij} = v_j$ or $v_{ij} = v_j$, contrary to the assumption that $C_0$ is a dicycle. This proves Claim I.

By Claim I, for every dicycle $C$ in $D$, all arcs in $C$ (except for the arc $(v_2, v_1)$) belong to exactly one of oriented graphs $D_n$ ($i = 1, 2, \ldots, n$). By Definition 4 and Lemma 6, every dicycle cover of oriented graph $D_n$ ($i = 1, 2, \ldots, n$) must have at least $m_i - n_i + 1$ dicycles. This completes the proof.

By Corollary 7 and Theorem 11, we have the following corollary.

**Corollary 12.** Let $D_{n1}, D_{n2}, \ldots, D_{ns}$ be $s$ disjoint strong tournaments with $n_1, n_2, \ldots, n_s$ vertices, respectively. Then $D_{n1} \oplus D_{n2} \oplus \cdots \oplus D_{ns}$ has a dicycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq (n_1(n_1 - 1)/2 + n_2(n_2 - 1)/2 + \cdots + n_s(n_s - 1)/2) - (n_1 + n_2 + \cdots + n_s) + s$. This bound is best possible.

Let $G_n$ be a Hamiltonian graph with $n$ vertices and $m$ arcs; let $D_n$ ($i$ is an integer) denote a Hamiltonian orientation of $G_n$. For a positive integer $s$, let $H(G_n, s)$ denote the family of all $2$-sum generated digraphs $D_n \oplus D_n \oplus \cdots \oplus D_n$, as well as a member in the family (for notational convenience). By the definition of $H(G_n, s)$, we have $H(G_n, 1) = D_n$ and $H(G_n, s) = H(G_n, s - 1) \oplus D_n$. The conclusions of the next corollaries follow from Theorem 1. The sharpness of these corollaries can be demonstrated using similar constructions displayed in Lemma 6 and Corollary 8.

**Corollary 13.** Let $m, n \geq 3$ be integer, let $G_n$ be a Hamiltonian graph with $n$ vertices and $m$ edges, and let $K_n$ be a complete graph on $n \geq 3$ vertices:

(i) Any member in $H(G_n, s)$ has a dicycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq s(m - n + 1)$. This bound is best possible.

(ii) In particular, any $H(K_n, s)$ has a dicycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq s(n(n - 1)/2 - n + 1)$. This bound is best possible.
Corollary 14. Let \( m, n \geq 3 \) be integer, let \( B_n \) be a Hamiltonian bipartite graph with \( 2n \) vertices and \( m \) edges, and let \( K_{n,n} \) be a complete bipartite graph:

(i) Any \( H(B_n, s) \) has a dicycle cover \( \mathcal{C} \) with \( |\mathcal{C}| \leq s(m - 2n + 1) \). This bound is best possible.

(ii) In particular, any \( H(K_{n,n}, s) \) has a dicycle cover \( \mathcal{C} \) with \( |\mathcal{C}| \leq s(n - 1)^2 \). This bound is best possible.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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