Regular matroids without disjoint circuits

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Abstract

A cosimple regular matroid $M$ does not have disjoint circuits if and only if $M \in \{M(K_{3,3}), M^*(K_n) \ (n \geq 3)\}$. This extends a former result of Erdős and Pósa on graphs without disjoint circuits.

Key words: regular matroid, disjoint circuits.

1 Introduction

We shall assume familiarity with graph theory and matroid theory. For terms that are not defined in this note, see Bondy and Murty [1] for graphs, and Oxley [3] or Welsh [6] for matroids. We allow graphs to have multiple edges but we forbid loops. To be consistent with the matroid terminology, a circuit in a graph is a nontrivial 2-regular connected subgraph, and a cycle is a disjoint union of circuits.

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If \( G \) is a graph and if \( V_1, V_2 \) are two disjoint vertex subsets of \( G \), then \( |V_1, V_2| \) denote the set of edges in \( G \) with one end in \( V_1 \) and the other end in \( V_2 \). For a vertex \( v \in V(G) \), let
\[
E_C(v) = \{ e \in E(G) : e \text{ is incident with } v \}.
\]

Let \( M \) and \( N \) denote two matroids. If \( \{e, f\} \) is a circuit of \( M^* \) and if \( M/f = N \), then \( M \) is a serial extension of \( N \). In this case, we say that \( f \) is serial to \( e \). Note that being serial is an equivalence relation on \( E(M) \) for a matroid \( M \). The corresponding equivalence classes are the serial classes of \( M \). Dually, two elements \( e, f \) are parallel in \( M \) if they are serial in \( M^* \); being parallel is an equivalence relation on \( E(M) \) and the equivalence classes are the parallel classes of \( M \). An equivalence class is nontrivial if it has more than one elements.

In 1960, Erdős and Pósa consider the problem of determining all connected graphs that do not have edge-disjoint circuits. We view the complete graph \( K_3 \) as a plane graph and let \( K_3^* \) denote the geometric dual of the plane graph \( K_3 \).

**Theorem 1.1** (Erdős and Pósa [2]) Let \( G \) be a graph with \( \delta(G) \geq 3 \). The following are equivalent.
(i) \( G \) does not have edge-disjoint circuits.
(ii) \( G \in \{K_{3,3}, K_3^*, K_4\} \).

Since a graph \( G \) does not have disjoint circuits if and only if any subdivision of \( G \) does not have disjoint circuits, the following corollary follows immediately.

**Corollary 1.2** (Erdős and Pósa [2]) Let \( G \) be a simple graph of order \( n \geq 3 \).
(i) If \( |E(G)| \geq n + 4 \), then \( G \) has 2 edge-disjoint circuits.
(ii) The graph \( G \) with \( |E(G)| = n + 3 \) does not have edge-disjoint circuits if and only if \( G \) can be obtained from a subdivision \( G_0 \) of \( K_{3,3} \) by adding a forest and exactly one edge, joining each tree of the forest to \( G_0 \).

Theorem 1.1 can be viewed as a result on cosimple graphic matroids. Thus we consider generalizing Theorem 1.1. to matroids. Our main results of this note are the following.

**Theorem 1.3** Let \( M \) be a connected cosimple regular matroid. The following are equivalent.
(i) $M$ does not have disjoint circuits.
(ii) $M \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$.

**Corollary 1.4** Let $M$ be a regular matroid. Then $M$ has no disjoint circuits if and only if one of the following holds:
(i) $M = U_{m,m}$, for some integer $m > 0$, or
(ii) $M$ is a serial extension of a member in $\{M(K_{3,3}), U_{0,1}\} \cup \{M^*(K_n), n \geq 3\}$, or
(iii) $M = M_1 \oplus M_2$ is the direct sum of two matroids $M_1$ and $M_2$, where $M_1$ is a serial extension of a member in $\{M(K_{3,3}), U_{0,1}\} \cup \{M^*(K_n), n \geq 3\}$ and where $M_2 \cong U_{m,m}$, for some $m = |E(M)| - |E(M_1)| \geq 1$.

## 2 Proof of the Main Result

We follow Seymour [5] to introduce the notion of binary matroid sums. Given two sets $X$ and $Y$, the symmetric difference of $X$ and $Y$, is

$$X \Delta Y = (X \cup Y) - (X \cap Y).$$

Let $M_1$ and $M_2$ be two binary matroids where $E(M_1)$ and $E(M_2)$ may intersect. Define $M_1 \Delta M_2$ to be the binary matroid on $E = E(M_1) \Delta E(M_2)$ whose cycles are the nonempty, minimal subsets of $E$ of the form $X_i \Delta X_j$, where for each $i = 1, 2$, $X_i$ is a disjoint union of circuits of $M_i$. The binary matroid sums are defined as follows.

(i) If $E(M_1) \cap E(M_2) = \emptyset$, then $M_1 \Delta M_2$ is the 1-sum of $M_1$ and $M_2$ (also referred as a direct sum).

(ii) If $E(M_1) \cap E(M_2) = \{e_0\}$, such that, for each $i \in \{1, 2\}$, the element $e_0$ is neither a loop nor a coloop of $M_i$, then $M_1 \Delta M_2$ is the 2-sum of $M_1$ and $M_2$.

(iii) If $E(M_1) \cap E(M_2) = C$, where $C$ is a 3-circuit of both $M_1$ and $M_2$, such that $C$ includes no cocircuit of either $M_1$ or $M_2$, and such that for $i \in \{1, 2\}$, $|E(M_i)| \geq 7$, then $M_1 \Delta M_2$ is the 3-sum of $M_1$ and $M_2$.

For $k = 1, 2, 3$, we also use $M_1 \bigoplus_k M_2$ to denote the $k$-sum of two matroids $M_1$ and $M_2$. If each of $M_1$ and $M_2$ is isomorphic to a proper minor of $M_1 \bigoplus_k M_2$, then we say that $M$ is a proper $k$-sum of $M_1$ and $M_2$. For the case $k=1$, we also use $M_1 \bigoplus M_2$ for $M_1 \bigoplus_1 M_2$ to denote the direct sum of $M_1$ and $M_2$. 

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Let $A$ denote the matrix below
\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1
\end{bmatrix},
\]
and let $R_{10}$ denote the binary matroid $M_2[A]$.

Seymour's regular matroid decomposition theorem can be applied to cosimple matroids in the following form.

**Theorem 2.1** (Seymour [4]) Let $M$ be a cosimple connected regular matroid. Then one of the following holds.
(i) $M$ is cosimple and graphic.
(ii) $M$ is cosimple and cographic.
(iii) $M$ is isomorphic to $R_{10}$.
(iv) For $i \in \{2, 3\}$, $M = M_1 \bigoplus_k M_2$ is the proper 2-sum or 3-sum of two cosimple regular matroids $M_1$ and $M_2$, where both $M_1$ and $M_2$ are isomorphic to proper minors of $M$.

The following lemma is straightforward.

**Lemma 2.2** Let $G$ be a graph. If $M(G)$ is cosimple, then $\delta(G) \geq 3$.

**Proof:** Note that any edge incident with a degree 1 vertex in $G$ must be a loop of $M^*(G)$, and that the edges incident with a degree 2 vertex in $G$ must be in a parallel class of $M^*(G)$. Since $M(G)$ is cosimple, $M^*(G)$ does not have loops or nontrivial parallel classes. Hence we must have $\delta(G) \geq 3$. □

**Proof of Theorem 1.3** We first show that Theorem 1.3(i) implies Theorem 1.3(ii), and so we assume the $M$ is a connected cosimple regular matroid with no disjoint circuits. By Theorem 2.1, one of the conclusions in Theorem 2.1 must hold.

If $M$ is graphic, then we may assume that for some connected graph $G$, $M = M(G)$. By Lemma 2.2, $\delta(G) \geq 3$. Since $G$ has no disjoint circuits, by Theorem 1.1, $G \in \{K_{3,3}, K^*_3, K_4\}$, and so Theorem 1.3(ii) holds.
If $M$ is cographic, then we may assume that for some graph $G$, $M = M^*(G)$, where $G$ is a connected graph with $n = r(M) + 1$ vertices. Since $M$ is cosimple, $G$ is a simple graph, and so $G$ is a spanning subgraph of $K_n$, the complete graph on $n$ vertices. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$. If $G \neq K_n$, then we may assume that $v_1v_2 \notin E(G)$. In this case, $E_G(v_1) \cap E_G(v_2) = \emptyset$, contrary to Theorem 1.3(i). Therefore, we must have $G = K_n$, and so $M \in \{M^*(K_n), n \geq 3\}$.

If $M$ is isomorphic to $R_{10}$, then it is well known that $R_{10}$ is a disjoint union of a 4-circuit and a 6-circuit, contrary to Theorem 1.3(i). Thus $M \cong R_{10}$ is impossible.

Now suppose that 2.1(iv) holds. We argue by induction on $|E(M)|$. Since any matroid with at most 3 elements must be graphic, we assume that $|E(M)| = n \geq 4$, and Theorem 1.3(ii) holds for any matroid $M$ satisfying Theorem 1.3(i) with $|E(M)| < n$.

Since Theorem 2.1(iv) holds, for some $i \in \{2, 3\}$, $M = M_1 \bigoplus_i M_2$ is the proper $i$-sum of two cosimple regular matroids $M_1$ and $M_2$, where both $M_1$ and $M_2$ are proper minors of $M$.

If one of $M_1$ or $M_2$ has two disjoint circuits, then by the definition of binary matroid sums, $M$ would also have disjoint circuits, contrary to Theorem 1.3(i). Therefore, for each $i$, $M_i$ does not have disjoint circuits. Since $M_i$ is a proper minor of $M$, by induction, $M_1, M_2 \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$.

If $i = 2$, then we may assume that $e_0 \in E(M_1) \cap E(M_2)$. By the definition of 2-sum and by the fact that $M_1, M_2 \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$, $\exists C_1 \in \mathcal{C}(M_1)$ and $C_2 \in \mathcal{C}(M_2)$ such that $e_0 \notin C_i$. It follows that $C_1 \cap C_2 = \emptyset$ and so Theorem 1.3(i) is violated. Thus this is impossible.

Now assume that $i = 3$, and $Z = E(M_1) \cap E(M_2)$ is a 3 element circuit of both $M_1$ and $M_2$. Recall that $M_1, M_2 \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$. By the definition of a 3-sum, for any $i \in \{1, 2\}$, $|E(M_i)| \geq 7$ and so $M_i \notin \{M^*(K_3), M^*(K_4)\}$. Since there is no 3-circuits in either $M(K_{3,3})$ or a $M^*(K_n)$ with $n > 4$, it is impossible that both $|Z| = 3$ and $Z \in \mathcal{C}(M_1) \cap \mathcal{C}(M_2)$. This contradiction shows that this case is also impossible.

Thus if Theorem 1.3(i) holds, then we must have $M \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$.
Conversely, suppose $M \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$. Since $K_{3,3}$ is a bipartite simple graph, any circuit of $K_{3,3}$ has length at least 4. Suppose that $K_{3,3}$ has two disjoint circuits $C_1$ and $C_2$, then since $K_{3,3}$ is 3-regular, we must have $V(C_1) \cap V(C_2) = \emptyset$, and so $6 = |V(K_{3,3})| \geq |V(C_1)| + |V(C_2)| \geq 8$, a contradiction. Hence $M(K_{3,3})$ cannot have disjoint circuits. Suppose that $M = M^*(K_n), n \geq 3$ and write $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Suppose that $C_1$ and $C_2$ are two circuits of $M^*(K_n)$. Then $C_1$ is an edge cut of $K_n$ and so $C_1 = \{V_1, V_2\}$, for some proper vertex subset $V_1 \subseteq V(G)$ and $V_2 = V(G) - V_1$. Similarly, $C_2 = \{W_1, W_2\}$, where $\emptyset \neq W_1 \subseteq V(G)$ and $W_2 = V(G) - W_1 \neq \emptyset$. We may assume that $v_1 \in V_1 \cap W_1$. If $V_2 \cap W_2 \neq \emptyset$, say $v_2 \in V_2 \cap W_2$, then $v_1v_2 \in C_1 \cap C_2$. If $V_2 \cap W_2 = \emptyset$, then we have $W_2 \subseteq V_1, V_2 \subseteq W_1$. Since $\emptyset \neq [V_2, W_2] \subseteq [V_2, V_1] = C_1$ and $\emptyset \neq [W_2, W_2] \subseteq [W_1, W_2] = C_2$, then $C_1 \cap C_2 \neq \emptyset$. This proves that $M^*(K_n)$ does not have disjoint circuits.

Proof of Corollary 1.4 It suffices to show, by induction on $|E(M)|$, that if $M$ has no disjoint circuits, then one of (i), (ii) and (iii) holds. Let $M$ be a regular matroid that does not have disjoint circuits.

We first assume that $M$ is connected. If $M$ has a loop or a coloop, then since $M$ is connected, we must have $M \in \{U_{0,1}, U_{1,1}\}$, and so Corollary 1.4 (i) or (ii) must hold. Thus we assume that $M$ is loopless and coloopless.

If $M$ is connected and cosimple, then by Theorem 1.3, $M$ is a member of $\{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$ and so Corollary 1.4(ii) holds. Otherwise, $M$ has nontrivial serial classes. Let $\{e_1, e_2\}$ be a pair of serial elements in $M$. Since the intersection of any circuit and any cocircuit in a matroid $M$ cannot have exactly one element, any circuit in $M$ containing $e_1$ must also contain $e_2$. This implies that $M$ has no disjoint circuits if and only if $M/e_2$ has no disjoint circuits. By induction, $M/e_2$ is a serial extension of a member in $\{M(K_{3,3}), U_{0,1}\} \cup \{M^*(K_n), n \geq 3\}$. Since $M$ is a serial extension of $M/e_2$, $M$ is also a serial extension of a member in $\{M(K_{3,3}), U_{0,1}\} \cup \{M^*(K_n), n \geq 3\}$.

Now suppose that $M$ is not connected. Then $M = M_1 \oplus M_2 \oplus \cdots \oplus M_k$, where $M_1, M_2, \ldots, M_k$ are connected components of $M$. If $\forall i, M_i$ contains no circuits, then Corollary 1.4(i) holds. Otherwise, since $M$ has no disjoint circuits, exactly one connected component, say $M_1$, has at least one circuit. It follows that $M_2 \oplus \cdots \oplus M_k \cong U_{n,n}$ and so Corollary 1.4 (iii) must hold.
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