Cycle covering of plane triangulations

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Abstract. Bondy conjectures that if $G$ is a 2-edge-connected simple graph with $n$ vertices, then at most $(2n - 1)/3$ cycles in $G$ will cover $G$. In this note, we show that if $G$ is a plane triangulation with $n \geq 6$ vertices, then at most $(2n - 3)/3$ cycles in $G$ will cover $G$.

1. Introduction

We follow the notation of Bondy and Murty [BM], except where noted otherwise. An edge $e$ of a graph $G$ is called a multiple edge if $G - e$ has an edge $f$ having the same ends as $e$, and in this case we say that $e$ is an extra edge of $G - e$ parallel to the edge $f$. Graphs may have multiple edges but loops are prohibited. Let $G$ be a graph. For $X \subseteq E(G)$, the contraction $G/X$ is the graph obtained from $G$ by identifying the ends of each edge in $X$ and then deleting the resulting loops. A collection $C$ of cycles in $G$ is called a cycle cover (CC) of $G$, if every edge of $G$ lies in at least one cycle in $C$. It is obvious that $G$ has a CC if and only if $G$ is 2-edge-connected. For a graph with $\kappa'(G) \geq 2$, define

$$cc(G) = \min\{|C| : C \text{ is a CC of } G\}.$$ 

In [B], Bondy raised the following conjecture.

Conjecture SCC: If $G$ is a simple 2-edge-connected graph with $n$ vertices, then

$$cc(G) \leq \frac{2n - 1}{3}.$$ 

If $C$ is a collection of cycles of $G$ and if every edge in $G$ lies in exactly 2 members of $C$, then $C$ is called a cycle double cover (CDC) of $G$. The eminent cycle double cover conjecture, due to Seymour [S1] and Szekeres [S2], says that every 2-edge-connected graph admits a CDC. The following conjecture is also posted by Bondy in [B].

Conjecture SCDC: If $G$ is a simple 2-edge-connected graph with $n$ vertices, then $G$ admits a CDC with at most $n - 1$ cycles.
Lemma 2.1 follows. Hence we may assume that $C_1^i \neq C_2^j$ and \{e_j\}. Thus

$$C_1^i \cup (C_2 - \{C_1^i, C_2^j\}) \cup \{C_2^j \cup C_1^i, C_2^j \cup C_2^i\}$$

Lemma 2.1 follows again.

$\Gamma$ of $G$. The vertices of attachment of $H$ in $G$, denoted by $x$ in $V(H)$ that are incident with some edges in $E(G)$ — denotes a graph obtained from $H$ by adding an extra edge of $H$.

We that $H = \Gamma_1$ or $H = \Gamma_1^+$ (see Figure 1) with an extra to an edge in $E(\Gamma_1) - \{v_1 v_2, v_2 v_3, v_3 v_1\}$, such that $H$ is $A_G(H) \subseteq \{v_1, v_2, v_3\}$. Let $e_1$ be an extra edge parallel to $v_2 v_3$. Let $V_H = V(H) - \{v_1, v_2, v_3\}$.

(1) let $G' = (G - V_H) + e_2$ and we have

$$cc(G) \leq cc(G') + 1. \quad (2)$$

(2) let $G'' = (G - V_H) + \{e_1, e_2\}$ and we have

$$cc(G) \leq cc(G'') + 1. \quad (3)$$

(i) first. Let $C$ be a CC of $G'$, and let $G \in C$ be a cycle con-

$-e_2 + (v_2 v_5, v_5 v_4, v_4 v_6, v_6 v_3, v_3 v_2).$ and $F = v_2 v_4, v_1 v_6 v_5 v_3 v_2.$

$\Gamma'$, $F'$} is a CC of $G$ and so (2) holds.

Similar and uses the fact that we can always assume that $e_1$ cycles of any CC of $G''$.

We that $H = \Gamma_i$ or $H = \Gamma_i^+$ (see Figures 1 and 2) with an extra to an edge of $E(\Gamma_i) - \{v_1 v_2, v_2 v_3, v_3 v_1\}$, $(2 \leq i \leq 4)$,

graph of $G$ with $A_G(H) \subseteq \{v_1, v_2, v_3\}$. Let $e_i$ be an extra $e_i, e_1, (1 \leq i \leq 2)$, and let $V_H = V(H) - \{v_1, v_2, v_3\}$.

Let $G' = G - V_H$ and we have

$$cc(G) \leq cc(G') + 2. \quad (4)$$

$= \Gamma_i^i$. If $e$ in not incident with $v_1$, then let $G'' = G - e$ is incident with $v_1$, then let $G'' = G - V_H + e_2$. In either

$$cc(G) \leq cc(G'') + 2.$$
Proof: We consider the following cases.

Case 1: \( i = 2 \).

Let \( C \) be a CC of \( G' \) and let \( C \) be a cycle in \( C \) that contains \( v_1, v_3 \). Let \( C' = C - v_1 v_3 + \{ v_1 v_4, v_4 v_5, v_5 v_6, v_6 v_7, v_7 v_8, v_8 v_1 \} \), let \( F_1 = v_1 v_3 v_5 v_7 v_1 \) and \( F_2 = v_1 v_2 v_4 v_6 v_1 \). Then \( (C - \{ C \}) \cup \{ C', F_1, F_2 \} \) is a CC of \( G \) and so (4) holds.

The proof for (5) is similar and uses the fact that we can assume that \( e_1 \) and \( v_1 v_3 \) are in distinct cycles of any CC of \( G' \).

Case 2: \( i = 3 \).

Let \( C \) be a CC of \( G' \) and let \( C_1, C_2 \) be cycles in \( C \) that contain \( v_1, v_3 \) and \( v_2, v_3 \), respectively. (It may happen that \( C_1 = C_2 \).) Let \( C'_1 = C_1 - \{ v_1 v_3 \} + \{ v_1 v_4, v_4 v_5, v_5 v_6, v_6 v_7 \} \), \( C'_2 = C_2 - \{ v_2 v_3 \} + \{ v_2 v_5, v_5 v_7, v_7 v_9, v_9 v_1 \} \), if \( C_1 = C_2 \), then \( C'_1 = C'_2 \) is obtained by replacing \( v_1, v_3 \) by the above two paths, respectively, and let \( F_1 = v_1 v_3 v_5 v_7 v_9 v_1 \) and \( F_2 = v_1 v_2 v_4 v_6 v_1 \). Thus \( (C - \{ C_1, C_2 \}) \cup \{ C'_1, F_1, F_2 \} \) is a CC of \( G \), and so (4) holds.

Suppose that \( H \) is \( \Gamma_5^* \) and \( e \) is not incident with \( v_1 \). Let \( C' \) be a CC for \( G' \) and let \( C_1, C_2 \) be defined as above and let \( C \) be the cycle in \( C \) containing \( e_1 \).

If \( E(C_2) \neq \{ e_1, v_2 v_3 \} \), then \( C_e \neq C_2 \). Since \( e \) is not incident with \( v_1 \), there is a \((v_2, v_3)\)-path \( P \in \Gamma_1 \) containing \( e \) such that the internal vertices of \( P \) are in \( V_H \). Thus we can define \( C'_e \) to be \( C_e - e_1 \) plus the \((v_2, v_3)\)-path \( P \), and define \( C'_1, C'_2, F_1, F_2 \) as above. It follows that \( (C - \{ C_1, C_2 \}) \cup \{ C'_1, C'_2, F_1, F_2 \} \) is a CC of \( G \) and so (5) holds.

Thus we assume that \( E(C_2) = E(C_e) = \{ e_1, v_2 v_3 \} \). Without loss of generality, we assume that \( P \) is not parallel to \( v_5 v_7 \). Let \( F_1 = v_1 v_4 v_6 v_7 v_2 v_5 v_1 \), \( F_2 = v_1 v_3 v_5 v_7 v_9 v_1 \), and let \( F_3 \) be any cycle containing both \( v_5 v_7 \) and \( e_1 \). Thus \( (C - \{ C_1, C_2 \}) \cup \{ C'_1, F_1, F_3, F_2 \} \) is a CC of \( G \), and so (5) holds.

The case when \( e \) is incident with \( v_1 \) can be shown similarly.

Case 3: \( i = 4 \).

Let \( C \) be a CC of \( G' \) and let \( C_1, C_2 \) be cycles in \( C' \) containing \( v_1, v_3 \) and \( v_2, v_3 \), respectively. (Possibly \( C_1 = C_2 \).) Let \( C'_1 = C_1 - \{ v_1 v_3 \} + \{ v_1 v_4, v_4 v_5, v_5 v_6, v_6 v_7 \} \) and \( C'_2 = C_2 - \{ v_2 v_3 \} + \{ v_2 v_5, v_5 v_7, v_7 v_9, v_9 v_1 \} \), and let \( F_1 = v_1 v_3 v_5 v_7 v_9 v_1 \), and \( F_2 = v_1 v_2 v_4 v_6 v_1 \). Then \( (C - \{ C_1, C_2 \}) \cup \{ C'_1, F_1, F_2 \} \) is a CC of \( G \) and so (4) holds.

The proof when \( H = \Gamma_5^* \) is similar to that for the Case of \( i = 3 \).

Lemma 2.4. Suppose that \( H = \Gamma_5 \) or \( H = \Gamma_5^* \) (see Figure 3) with an extra edge \( e \) that is parallel to an edge \( d \) that \( d \) is parallel to an edge \( e \) of \( G' \) such that \( H \) is a subgraph of \( G' \) with \( A_G(H) \subseteq \{ v_1, v_2, v_3 \} \). Let \( V_H = V(G') - \{ v_1, v_2, v_3 \} \) and let \( G' = G - V_H \). Then

\[
cc(G) \leq cc(G') + 3.
\]
\( \text{cc}(G) \leq \text{cc}(G_2) + 1. \) \hspace{1cm} (9)

\( L''_k \) (see Figure 6) and let \( e' \notin E(G) \) be an extra edge between \( H \) and \( H' \). If \( H = L_1 \) or \( H = L_1' \) with an extra edge \( e \) that is parallel to \( H \), then \( \text{cc}(G) \leq \text{cc}(G_2) + 2. \) \hspace{1cm} (10)

8. If \( H = L''_1 \) or \( H = L''_1' \) with an extra edge \( e \) parallel to \( H \), then \( \text{cc}(G) \leq \text{cc}(G_4) + 2. \) \hspace{1cm} (11)

5. \( x, z, y \) and \( L''_1 = L''_1' \). Suppose that \( H \in \{ L''_1, L''_1' \} \) with an extra edge \( e \) parallel to an edge in \( G \), and with \( A_G(H) \subseteq \{ x, y, z \} \). Then \( \text{cc}(G) \leq \text{cc}(G_2) + 2. \) \hspace{1cm} (12)

\( x_1 \) to \( x_4x_5 \) if \( e \) is parallel to \( x_1x_2 \) or \( x_2x_3 \) or \( x_3x_4 \) or \( x_4x_5 \); or where \( e' \) is parallel to \( x_1 \), or \( x_2 \), or \( x_3 \), or \( x_4 \). In any case, we have \( \text{cc}(G) \leq \text{cc}(G_2) + 1. \) \hspace{1cm} (9)

\( C \to \{ C_1, C_2 \} \) containing \( x_2x_4 \) and still denote the resulting collection by \( C \to \{ C_1, C_2 \} \), for convenience. Thus \( (C \to \{ C_1, C_2 \}) \cup \{ C_2, C', F \} \) is a CC of \( G \) and so (9) must hold.

If \( C_1 \neq C_3 \), then let \( C'_1 = C_2 - x_2x_4 + x_2x_3x_4 \) and let \( C'_3 = C_3 - x_2x_3 + x_2x_3x_2, \) and let \( F' = x_1x_2x_2x_3x_4x_1x_1x_2x_3 \). Thus \( (C \to \{ C_2, C_3 \}) \cup \{ C'_2, C'_3 \} \) is a CC of \( G \). Hence (9) must hold again.

When \( H = L_1 \), we can replace \( e' \) in the cycle containing \( e' \) by a path in \( H \) containing the multiple edge and so (9) holds again.

To show (iii), we let \( C \) be a CC of \( G \) and let \( C_1, C_2, C_3, C_4 \) be cycles that contain \( e, e', x_2x_4, x_2x_3, x_2x_4, \) and \( x_2x_3x_4 \) respectively.

Assume that \( H = L''_1 \) or \( L''_1' \). Then \( C_4 \neq C_3 \) and \( C_4 \neq C_2 \). Let \( C_1 = C_1 - x_2x_4 + x_2x_3x_4, C'_2 = C_2 - x_2x_3 + x_2x_3x_2, \) and \( C'_4 = C_4 - x_2x_3x_4 + x_2x_3x_1, \) and let \( F' = \) a cycle containing \( x_2x_4 \) and the multiple edge (if it exists). Thus \( (C \to \{ C_1, C_2, C_3, C_4 \}) \cup \{ C'_2, C'_3, F' \} \) is a CC of \( G \) and so (10) holds also.

The proof for the case when \( H \in \{ L''_1, L''_1', L''_2, L''_2' \} \) is similar to that for \( H \in \{ L''_1, L''_2 \} \) and the proof for (iv) is similar to that for (iii). Thus they are omitted.

We shall show (v) for \( H \in \{ L''_1, L''_2 \} \). The proof for \( H \in \{ L''_1, L''_2' \} \) is similar. Let \( v \) denote the vertex in \( G \) to which \( x_2x_4 \) is contracted. Let \( C \) be a CC of \( G' \) and let \( C_4, C_3 \) be cycles in \( C \) containing \( e' \) and \( x_2x_4 \), respectively. (If \( H = L''_1 \), then just take \( C_3 \)).

If \( C_n = C_3 \), then let \( F' = x_1x_2x_3x_4x_2x_3x_4x_1x_2x_3x_4x_2 \) and let \( F'' \) be a cycle that contains the multiple edge \( e \). Thus \( (C \to \{ C_3 \}) \cup \{ F', F'' \} \) is a CC of \( G \) and so (12) holds.

Thus we assume that \( C_n \neq C_3 \). Let \( C_1 = G(E(C_1) - x_2x_4) \). Thus either \( C_1 \neq C_3 \) or \( C_1 \neq C_4 \) is a cycle in \( G \). Note that any cycle in \( C \to \{ C_4 \} \) can easily be adjusted to cycles in \( G \) (still denoted by \( C \to \{ C_4 \} \), for convenience).

Let \( F_1 = x_1x_2x_3x_4x_3x_4x_2x_3x_4x_3x_4x_2, \) and let \( C_1 \) be obtained from \( C_1 \) by replacing \( e' \) by \( e \) (\( x_2x_3x_2, x_4x_3, x_3x_4, x_4x_2 \)). Then \( C_1 \) is a cycle in \( G \), then let \( C_1' = C_1 + x_2x_3x_2x_3 \) and let \( C_1' \) be a cycle in \( G \), then let \( C_1'' = C_1' + x_2x_3x_2x_3 \). Thus in any case, \( (C \to \{ C_3, C_n \}) \cup \{ C_1', C_2', F_1 \} \) is a CC of \( G \) and so (12) holds.

Let \( C \) be a cycle of a plane graph \( G \). Define \( \text{Int}(C) \) to be the vertices of \( C \) inside (exclusively) \( G \). Define \( \text{Ext}(C) \) similarly. The cycle \( C \) is trivial if \( \text{Int}(C) = \emptyset \) and is acyclic if the underlying simple graph of \( G[\text{Int}(C)] \) is acyclic.

A k-face of a plane graph \( G \) is a face of degree \( k \). Define \( L(n) \) as the graph in Figure 5.

**Lemma 2.7**. Let \( G \) be a plane triangulation with \( \mu(G) = 1 \) and with \( n = |V(G)| \geq 3 \). If the exterior face of \( G \) is a 2-cycle, and if \( C \) is acyclic, then
Proof: Let $v_1, v_2$ be the two vertices in $V(C)$ and let $e_1, e_2$ be the two edges in $E(C)$. Since $\mu(G) \leq 1$, and since $G$ is a plane triangulation, $e_1$ must lie in a 3-face $C$ inside $G$. Let $v_3$ be the vertex in $V(C_{i}) - \{v_1, v_2\}$. If $v_3$ has degree at least 4, then since $G$ is a triangulation, $v_3$ and two of its neighbors other than $v_1, v_2$ would form a 3-cycle inside $G$, contrary to the assumption that $G$ is acyclic. If $v_3$ has degree 2, then we have $n = 3$ and $G = L(3)$. Hence $v_3$ has degree 3, and so $G - v_3$ is also a plane triangulation with $C$ as an acyclic exterior face. Thus by induction, $G - v_3 = L(n - 1)$ and so $G = L(n)$.

Lemma 2.8. Let $G$ be a simple plane triangulation. If the exterior face of $G$ is a 3-cycle $C$ and if $C$ is acyclic, then either $G$ contains a subgraph $H \in \{L_6, \Gamma_2\}$ (using the notation in Figure 4) with $Ac(H) \subseteq \{x_1, x_2, x_3, x_4\}$ or $G = \Gamma(n)$, where $n = |V(G)|$.

Proof: Suppose $G = v_1, v_2, v_3, v_4$. Since $G$ is a plane triangulation, $v_2, v_3$ lies in a 3-face $C = v_2, v_3, v_2$ with $v \in Int(C)$. Let $v_2 = u_1, u_2, \ldots, u_m = v_3$ be the neighbors of $v$ in $G$ such that they are ordered clockwise by the planar embedding of $G$.

Since $G$ is a simple plane triangulation, $v_2, v_3, v_2, v_3, v_2, \ldots$, must be 3-faces. Since $C$ is an acyclic, either $m = 3$, or $4 \leq m \leq 5$ and $v_3 = v_1$.

If $m = 5$ and $u_1 = v_2$, then $G$ contains a subgraph $H = \Gamma_6$ with $x_1 = v_2, x_2 = v_3, x_3 = v_1, x_4 = v_2$. If $m = 4$ and $u_3 = v_4$, then, since $u_1, u_2, u_3, u_4$ is a 3-face, $G = \Gamma_4$. Hence we may assume that $m = 3$. If $u_2 = v_1$, then $G = \Gamma_4$. Thus we assume that $x_2 \neq x_1$, and so $G - v$ is also a plane triangulation with $C$ as the exterior face. By induction, Lemma 2.8 holds.

Lemma 2.9. Suppose that $G$ is a plane graph, and that $G$ has a nontrivial 2-cycle $C$ with $V(C) = \{v_1, v_2, v_3\}$ and $E(C) = \{e_1, e_2\}$. Let $H = G - ExtC$.

(i) If $H = L(3)^+$ such that the extra edge $e$ is parallel to $v_1, v_3$, then letting $G' = G / v_3, v_2$, we have $cc(G) \leq cc(G')$.

(ii) If $H = L(4)$, then letting $G' = G - IntC$, we have $cc(G) \leq cc(G') + 1$.

(iii) If $H = L(4)^+$ such that the extra edge $e$ is not parallel to any of $\{e_1, e_2\}$, then letting $e'$ be an extra edge parallel to $e_1$ and let $G'' = G - IntC + e'$, we have $cc(G) \leq cc(G'') + 1$.

(iv) If $H$ is isomorphic to $\Gamma(5)^+$, such that the exterior face of $H$ is $C$, then letting $G' = G - IntC$, we have $cc(G) \leq cc(G') + 2$.

Proof: (i) of Lemma 2.9 is trivial. We now show (ii). Let $C$ be a CC of $G'$ and let $C_1$ be a cycle in $C$ containing $e_1$. Define $C'_1 = C_1 - e_1 + \{v_1, v_4, v_5, v_6\}$ and $F = G[C'_1 - v_6]$. Thus $C' = \{C_1, F\}$ is a CC of $G$ and so (ii) of Lemma 2.9 holds.

Now we show (iii). Let $C$ be a CC of $G'$. Note that $\{e_1\} = \{e_1, e_2, e'\}$ in $G'$ this time. We may assume that $e_1$ and $e'$ are in distinct cycles $C_1$ and $C_2$.

respectively. Define $C'_1$ and $F$ as above. Since $e$ is not parallel to $e_1$, there is a $(v_1, v_2)$-path $P$ in $H - \{e_1, e_2\}$ containing $e$. Define $C'_e = C_e - e' + P$. Thus $C = \{C_1, C_2, C'_e\}$ is a CC of $G$.

Now we show (iv). Let $C$ be a CC of $G'$ and let $C_1$ be a cycle in $C$ containing $e_1$, $1 \leq i \leq 2$. Note that no matter where $e_1, e_2$ lie in $H$, $H - \{e_1, e_2\}$ has a spanning cycle and so $H - \{e_1, e_2\}$ has two internally disjoint $(v_1, v_2)$-paths $P_1$ and $P_2$. Let $C_i = C_i - e_i + P_i$. $(1 \leq i \leq 2)$. (When $C_1 = C_2 = C$, let $C_i = P_i$.) Thus it is easy to see that the edges in $H - E(P_1) \cup E(P_2)$ can be covered by two cycles in $H$ and so (iv) follows.

Define plane graphs $\Gamma^4, \Gamma^5, \Gamma^6$ as the graphs in Figure 6.

Lemma 2.10. Let $G$ be a plane triangulation with $\mu(G) \leq 1$ and with $4 \leq |V(G)| \leq 5$. If the exterior face of $G$ is a 3-face, then $G$ is isomorphic to one graph in $\{\Gamma_4, \Gamma_5, \Gamma_6, L(6)^+, \Gamma_6\}$.

Proof: The proof is straightforward.

For each $i$, $(1 \leq i \leq 5)$, define $\Gamma_i'$ to be the simple plane triangulation obtained from $\Gamma_i$ by adding a new vertex $v_0$ in the exterior face of $\Gamma_i$ and by joining $v_0$ to each of $v_1, v_2, v_3$ with a new edge, respectively.

Lemma 2.11: If $G$ is isomorphic to one of the graphs below,

$$\{\Gamma_i, \Gamma_i', \Gamma_i^+, (\Gamma_i')^+, (1 \leq i \leq 5), \Gamma(6), \Gamma(6)^+, L(6), L(6)^+, \Gamma_6, \Gamma_6^+, \Gamma_6\}$$

Then $cc(G) \leq \frac{|V(G)| - 3}{3} + \frac{\mu(G)}{2}$.

Proof: The proof is routine and so is omitted.

3. The Proof of Theorem 1.2

We argue by contradiction and assume that $G$ is a counterexample to Theorem 1.2 such that $|V(G)| + \mu(G)$ is as small as possible, and subject to (14), $|E(G)|$ is minimized.

If $G$ has two 2-faces, then we pick two distinct edges, $e, e'$ (say), from each of these 2-faces. Thus $\mu(G) = \mu(G - \{e, e', \}) + 2$ and so by (14) and (15), $G$ is not a counterexample, contrary to (13). Hence we assume that $G$ has at most one 2-face.
and Lemma 3.1, $G$ must have two edges $e_1$ with $[e_1] \neq [e_2]$ such that, for $G'' = G - \{e_1', e_2\}$, $G$ has a cycle containing both $(15)$,

$$\frac{3}{2} + \frac{\mu(G) - 2}{2} + 1,$$

as is forbidden in $G$:

$H \subseteq \{v_1, v_2\}$,

$\{v_1, v_2, v_3\}$, $(1 \leq i \leq 5)$,

$L_6, L_6', L_6''$ with $A_G(H) \subseteq \{x_1, x_2, x_3\}$,

defined as in Lemma 2.2. If $|V(G')| = 4$. If $|V(G')| \geq 6$, then by (14) and

$$\frac{3}{2} + \frac{\mu(G)}{3} + 1,$$

section. Thus by $|V(G)| \geq 6$, we have one can easily check that $G$ is not a

by using reduction lemmas in section

triangulation, the exterior face of $G$

$\geq 6$, $G$ must have a nontrivial 3-cyclic, then in particular, the exterior

edge from the exterior 2-face is also

8, $G$ must contain either $\Gamma_6$, $\Gamma_6$, or a cyclic 3-cycle. Let $C_0$ be a cyclic

(19)

$C_0$ is either trivial or acyclic. By

cycle in $G[\text{Int}C_0]$, then

(20)
so that the remaining edge in \( E(C) \), if there is any, is covered by \( u_4 u_2 u_5 u_3 u_6 u_1 \). Then \( \{ L' : L \in \mathcal{C} \} \cup \{ F \} \) is a CC of \( G \) and so \( \mu(G) \geq \max \{ (G) - \text{cc}(G_a) - 1 \} \geq \frac{2(n-2)-3}{3} + \frac{\mu(G)}{2} + 1 \), then by (14), (22) and (23),

\[
\mu(G) \leq \text{cc}(G_a) + 1 \leq \frac{2(n-2)-3}{3} + \frac{\mu(G)}{2} + 1,
\]

5, then since \( |V(C) \cap V(C_0)| = 0 \) and since \( u \) is a vertex of \( G \) in \( G_a \), it follows by Lemma 2.10 that \( G \in \{ \Gamma_1, \Gamma_1^+, \Gamma_{1'} \} \). In 2.11, \( G \) is not a counterexample, either.

3. The proofs for these subcases are similar to that when \( k = 0 \).

ne \( i \neq j \), \( E(C_i) \cap E(C_j) \neq \emptyset \).

(CJ \( \neq \emptyset \), for every \( i \neq j \), then \( u_4 = u_5 = u_6 \), contrary to Lemma 3.3. Hence we assume that

\[
E(C_3) \cap (E(C_1) \cup E(C_2)) = \emptyset \text{ and } u_4 = u_5.
\]

(25)

\( \neq \emptyset \) and \( \text{Int} C_2 \neq \emptyset \) or \( |\text{Int} C_3| \geq 1 \) and \( \text{Int} C_1 = \text{Int} C_2 = \emptyset \).

\( \cup \{ u_1, u_2, u_3, u_4 \} \} \) contains a subgraph isomorphic to \( \{ L_{16, L_{16}^+} \} \), contrary to Lemma 3.3. Thus we assume that

\[
\text{Int} C_1 = \emptyset.
\]

(26)

= 0 and \( |\text{Int} C_2| > 0 \). Then \( G \) has a forbidden subgraph \( H \) one of \( \{ L_6, L_6^+ \} \). This case can be excluded by applying (v) of

\[
\text{Int} C_2 = \text{Int} C_3 = \emptyset \text{ and } u_4 \in \text{Int} C_0.
\]

2.1 triangulation, there are \( u_7 \) and \( u_8 \) in \( V(G) \), such that \( C_4 = C_5 = u_7 u_8 u_4 u_3 \) are 3-cycles satisfying (21). Applying the previous 3-cycles \( C_4 \) and \( C_5 \), we conclude that \( \text{Int} C_4 = \text{Int} C_5 = \emptyset \).

\( \cup \{ u_1, u_2, u_3, u_4 \} \) and let \( G_5 = (G - \{ u_1 u_4, u_5 u_4, u_8 u_3 \})/E(H) \).

oof for (24), we can similarly show first that

\[
\mu(G_h) \leq \mu(G) \text{ and } \text{cc}(G) \leq \text{cc}(G_h) + 2,
\]

(27)

is not a counterexample, contrary to (13).

\( \text{Int} C_2 = \text{Int} C_3 = \emptyset \text{ and } u_4 \notin \text{Int} C_0. \)
Thus $u_4 \in V(C_0)$. It follows from Case 1 and Cases (2A) - (2C) that for any trivial 3-cycle $C' = z_1 z_2 z_3 z_4$ in $G \setminus \text{int}(C_0)$, there must be $z_4, z_5 \in V(C_0)$ such that $z_4 z_2 z_1 z_2 z_4 z_5 z_3 z_4 z_1 z_3 z_5 z_1$ are trivial 3-faces in $M$, with $z_1 = u_4$ $(1 \leq i \leq 4)$ and $z_5 = u_6$. Note that $C' = z_1 z_2 z_1 z_4 z_2 z_3 z_1$ must be a trivial 3-face since otherwise $G$ contains a $L_6$, contrary to Lemma 3.3. Call $G(\{z_1, z_2, z_3, z_4, z_5\})$ an associated $\Gamma(5)$ with edge $z_4 z_5 \in E(C_0)$. For each edge in $E(C_0)$, there is at most one associated $\Gamma(5)$ with the given edge. Delete $z_1, z_2$ from the associated $\Gamma(5)$ with $z_4 z_5$, and do the same for other associated $\Gamma(5)$'s with other edges in $E(C_0)$, (if there are any). Then the resulting graph is again a triangulation in which $C_0$ is an acyclic 3-cycle, and so by Lemmas 7 and 8, either $M$ contains a trivial 3-cycle that satisfies Case 1 or one of Cases (2A) - (2C), or $G$ contains $L_6$ or $\Gamma(6)$, or $M - E(C_0)$ is isomorphic to the graph $L_{11}$ in Figure 8.

Thus we may assume that $M - E(C_0) \cong L_{11}$. Let $G_e = G - \{z_8, z_9, z_{10}\}$. Then it is easy to see that

$$\text{cc}(G) \leq \text{cc}(G_e) + 2.$$ \hspace{1cm} (28)

Thus by (14) and since $|V(G_e)| \geq 7$, $G$ must satisfy (1), contrary to (13).

Since every case leads to a contradiction, Theorem 1.2 is proved.

References


