Group Connectivity of Graphs

Hong-Jian Lai

Department of Mathematics, West Virginia University
Contents:

- Notations and Definitions
- Nowhere Zero Flows: Conjectures and Progresses
Contents:

- Notations and Definitions
- Nowhere Zero Flows: Conjectures and Progresses
- Group Connectivity of Graphs: The Nonhomogeneous Case
Contents:

- Notations and Definitions
- Nowhere Zero Flows: Conjectures and Progresses
- Group Connectivity of Graphs: The Nonhomogeneous Case
- Planar Graphs, Line Graphs, Highly Connected Graphs
Contents:

- Notations and Definitions
- Nowhere Zero Flows: Conjectures and Progresses
- Group Connectivity of Graphs: The Nonhomogeneous Case
- Planar Graphs, Line Graphs, Highly Connected Graphs
- Chordal Graphs, Graphs with Small Diameter.
Contents:

- Notations and Definitions
- Nowhere Zero Flows: Conjectures and Progresses
- Group Connectivity of Graphs: The Nonhomogeneous Case
- Planar Graphs, Line Graphs, Highly Connected Graphs
- Chordal Graphs, Graphs with Small Diameter
- Triangulated Graphs
Contents:

- Notations and Definitions
- Nowhere Zero Flows: Conjectures and Progresses
- Group Connectivity of Graphs: The Nonhomogeneous Case
- Planar Graphs, Line Graphs, Highly Connected Graphs
- Chordal Graphs, Graphs with Small Diameter
- Triangulated Graphs
- Disproof of Barat-Thomassen Conjecture
Notation:

- $G$: a graph, with vertex set $V = V(G) = \{v_1, v_2, \cdots, v_n\}$, and edge set $E = E(G) = \{e_1, e_2, \cdots, e_m\}$. 

$D(G)$: an orientation of $G$.

$D = (d_{ij})_{n \times m}$: vertex-edge incidence matrix, where $d_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is oriented away from } v_i \\ 1 & \text{if } e_j \text{ is oriented into } v_i \\ 0 & \text{otherwise} \end{cases}$
Notation:

- $G$: = a graph, with vertex set $V = V(G) = \{v_1, v_2, \cdots, v_n\}$, and edge set $E = E(G) = \{e_1, e_2, \cdots, e_m\}$.
- $D(G)$: = an orientation of $G$. 
Notation:

- $G$: a graph, with vertex set $V = V(G) = \{v_1, v_2, \cdots, v_n\}$, and edge set $E = E(G) = \{e_1, e_2, \cdots, e_m\}$.

- $D(G)$: an orientation of $G$.

- $D = (d_{ij})_{n \times m}$: vertex-edge incidence matrix, where

$$d_{ij} = \begin{cases} 
1 & \text{if } e_j \text{ is oriented away from } v_i \\
-1 & \text{if } e_j \text{ is oriented into } v_i \\
0 & \text{otherwise}
\end{cases}$$
Notation

- $A$: an abelian (additive) group with identity $0$, and with $|A| \geq 3$ and $A^* = A - \{0\}$. 
Notation

- $A$: an abelian (additive) group with identity $0$, and with $|A| \geq 3$ and $A^* = A - \{0\}$.
- $F(G, A) = \{f : E \mapsto A\}$, and $F^*(G, A) = \{f : E \mapsto A^*\}$. 
Notation

- $A$: an abelian (additive) group with identity $0$, and with $|A| \geq 3$ and $A^* = A - \{0\}$.

- $F(G, A) = \{f : E \mapsto A\}$, and $F^*(G, A) = \{f : E \mapsto A^*\}$.

- A function $f : E \mapsto A$ can be viewed as an $m$-dimensional vector

\[ f = (f(e_1), f(e_2), \cdots, f(e_m))^T. \]
Notation

- $A$: an abelian (additive) group with identity 0, and with $|A| \geq 3$ and $A^* = A - \{0\}$.
- $F(G, A) = \{ f : E \mapsto A \}$, and $F^*(G, A) = \{ f : E \mapsto A^* \}$.
- A function $f : E \mapsto A$ can be viewed as an $m$-dimensional vector
  $$f = (f(e_1), f(e_2), \cdots, f(e_m))^T.$$  
- A function $b : V \mapsto A$ can be viewed as an $n$-dimensional vector
  $$b = (b(v_1), b(v_2), \cdots, b(v_n))^T.$$
Nowhere-zero $A$-flows (or $A$-NZFs)

- **Assumption:** For any graph $G$, we assume that a fixed orientation $D(G)$ of $G$ is given.

- **Notation:** $\forall a \in A$, $1 \cdot a = a$, $(-1) \cdot a = -a$ (additive inverse of $a$ in $A$), and $0 \cdot a = 0$ (additive identity of $A$)
Nowhere-zero $A$-flows (or $A$-NZFs)

- **Assumption:** For any graph $G$, we assume that a fixed orientation $D(G)$ of $G$ is given.

  **Notation:** $\forall a \in A$, $1 \cdot a = a$, $(-1) \cdot a = -a$ (additive inverse of $a$ in $A$), and $0 \cdot a = 0$ (additive identity of $A$)

- For any $f \in F(G, A)$, the boundary of $f$ is $\partial f := Df$. That is, $\forall v_i \in V, \partial f(v_i) = Df(v_i)$, which is the $v_i$th component of the vector $Df$. 
Nowhere-zero $A$-flows (or $A$-NZFs)

- **Assumption:** For any graph $G$, we assume that a fixed orientation $D(G)$ of $G$ is given.

  **Notation:** $\forall a \in A$, $1 \cdot a = a$, $(-1) \cdot a = -a$ (additive inverse of $a$ in $A$), and $0 \cdot a = 0$ (additive identity of $A$)

- For any $f \in F(G, A)$, the boundary of $f$ is $\partial f := Df$. That is, $\forall v_i \in V$, $\partial f(v_i) = Df(v_i)$, which is the $v_i$th component of the vector $Df$.

- A function $f \in F^*(G, A)$ is a nowhere-zero $A$-flow (or just an $A$-NZF) if $Df = 0$ (the all zero vector).
Integer Flows

- $\mathbb{Z}$: the abelian group of integers.

A function $f : \mathbb{Z}^k \rightarrow \mathbb{Z}$ is a nowhere-zero $k$-flow (or just a $k$-NZF) if $\nabla f = 0$, and if $0 < |f(e)| < k$.

Tutte: If $G$ has a $k$-NZF, then $G$ has a $(k+1)$-NZF.

Tutte: A graph $G$ has an $A$-NZF if and only if $G$ has an $|A|$-NZF.
Integer Flows

- $\mathbb{Z}$: the abelian group of integers.
- $\mathbb{Z}_k$: the abelian group of mod $k$ integers.
Integer Flows

- \( \mathbb{Z} \): = the abelian group of integers.

- \( \mathbb{Z}_k \): = the abelian group of mod \( k \) integers.

- A function \( f \in F^*(G, \mathbb{Z}) \) is a nowhere-zero \( k \)-flow (or just a \( k \)-NZF) if \( Df = 0 \), and if \( \forall e \in E(G) \), \( 0 < |f(e)| < k \).
Integer Flows

- $\mathbb{Z}$: = the abelian group of integers.
- $\mathbb{Z}_k$: = the abelian group of mod $k$ integers.
- A function $f \in F^*(G, \mathbb{Z})$ is a nowhere-zero $k$-flow (or just a $k$-NZF) if $Df = 0$, and if $\forall e \in E(G)$, $0 < |f(e)| < k$.
- Tutte: If $G$ has a $k$-NZF, then $G$ has a $(k + 1)$-NZF.
Integer Flows

- $\mathbb{Z}$: the abelian group of integers.
- $\mathbb{Z}_k$: the abelian group of mod $k$ integers.
- A function $f \in F^*(G, \mathbb{Z})$ is a nowhere-zero $k$-flow (or just a $k$-NZF) if $Df = 0$, and if $\forall e \in E(G)$, $0 < |f(e)| < k$.
- Tutte: If $G$ has a $k$-NZF, then $G$ has a $(k + 1)$-NZF.
- Tutte: A graph $G$ has an $A$-NZF if and only if $G$ has an $|A|$-NZF.
Some Properties

- If some orientation $D(G)$ has an $A$-NZF or a $k$-NZF, then for any orientation of $G$ also has the same property, and so having an $A$-MZF or a $k$-NZF is independent of the choice of the orientation.
Some Properties

- If some orientation $D(G)$ has an $A$-NZF or a $k$-NZF, then for any orientation of $G$ also has the same property, and so having an $A$-MZF or a $k$-NZF is independent of the choice of the orientation.

- If for an abelian group $A$, a connected graph $G$ has an $A$-NZF, then $G$ must be 2-edge-connected. (That is, $G$ does not have a cut edge).
Some Properties

- If some orientation $D(G)$ has an $A$-NZF or a $k$-NZF, then for any orientation of $G$ also has the same property, and so having an $A$-MZF or a $k$-NZF is independent of the choice of the orientation.

- If for an abelian group $A$, a connected graph $G$ has an $A$-NZF, then $G$ must be 2-edge-connected. (That is, $G$ does not have a cut edge).

- We shall only consider 2-edge-connected graphs $G$ and define

$$\Lambda(G) = \min\{k : G \text{ has a } k\text{-NZF}\}.$$
Tutte’s Conjectures

(5-flow) Every 2-edge-connected graph has a 5-NZF.
Tutte’s Conjectures

- (5-flow) Every 2-edge-connected graph has a 5-NZF.
- (4-flow) Every 2-edge-connected graph without a subgraph contractible to $P_{10}$, the Petersen graph, must have a 4-NZF.
Tutte’s Conjectures

- (5-flow) Every 2-edge-connected graph has a 5-NZF.
- (4-flow) Every 2-edge-connected graph without a subgraph contractible to $P_{10}$, the Petersen graph, must have a 4-NZF.
- (3-flow) Every 4-edge-connected graph has a 3-NZF.
Tutte’s Conjectures

- (5-flow) Every 2-edge-connected graph has a 5-NZF.
- (4-flow) Every 2-edge-connected graph without a subgraph contractible to $P_{10}$, the Petersen graph, must have a 4-NZF.
- (3-flow) Every 4-edge-connected graph has a 3-NZF.
- (Jaeger’s weak 3-flow conjecture) There exists an integer $k > 0$ such that every $k$-edge-connected graph has a 3-NZF.
Nowhere zero flows and colorings

Tutte: For a plane graph $G$, $G$ has a face $k$-coloring if and only if $G$ has a $k$-NZF.
Nowhere zero flows and colorings

- Tutte: For a plane graph $G$, $G$ has a face $k$-coloring if and only if $G$ has a $k$-NZF.

- These conjectures are theorems when restricted to planar graphs (need 4 Color Theorem for the 4-flow conjecture).
What do we know?

- Jaeger (1979, JCT(B)): Every 2-edge-connected graph has a 8-NZF.
What do we know?

- Jaeger (1979, JCT(B)): Every 2-edge-connected graph has a 8-NZF.
- Jaeger (1979, JCT(B)): Every 4-edge-connected graph has a 4-NZF.
What do we know?

- Jaeger (1979, JCT(B)): Every 2-edge-connected graph has a 8-NZF.
- Jaeger (1979, JCT(B)): Every 4-edge-connected graph has a 4-NZF.
- Seymour (1980, JCT(B)): Every 2-edge-connected graph has a 6-NZF.
What do we know?

- Jaeger (1979, JCT(B)): Every 2-edge-connected graph has a 8-NZF.
- Jaeger (1979, JCT(B)): Every 4-edge-connected graph has a 4-NZF.
- Seymour (1980, JCT(B)): Every 2-edge-connected graph has a 6-NZF.
- The 5-flow conjecture and 3-flow conjecture have also been verified for projective planes and some other surfaces.
What do we know?

- Robertson, Sanders, Seymour, Thomas (2000): Every 2-edge-connected cubic graph without a subgraph contractible to the Petersen graph has a 4-NZF.

- C. Q. Zhang and HJL (1992, DM): Every $4\log_2(j_V(G))$-edge-connected graph has a 3-NZF.

- Y. Shao, H. Wu, J. Zhou and HJL (2008, JCT(B)): Every $3\log_2(j_V(G))$-edge-connected graph has a 3-NZF.

- Z. H. Chen, H. Y. Lai and HJL (2002, DM): Tutte's flow conjectures are valid if and only if they are valid within line graphs.
What do we know?

- Robertson, Sanders, Seymour, Thomas (2000): Every 2-edge-connected cubic graph without a subgraph contractible to the Petersen graph has a 4-NZF.

- C. Q. Zhang and HJL (1992, DM): Every $4\log_2(|V(G)|)$-edge-connected graph has a 3-NZF.
What do we know?

- Robertson, Sanders, Seymour, Thomas (2000): Every 2-edge-connected cubic graph without a subgraph contractible to the Petersen graph has a 4-NZF.

- C. Q. Zhang and HJL (1992, DM): Every \(4 \log_2(|V(G)|)\)-edge-connected graph has a 3-NZF.

- Y. Shao, H. Wu, J. Zhou and HJL (2008, JCT(B)): Every \(3 \log_2(|V(G)|)\)-edge-connected graph has a 3-NZF.
What do we know?

- Robertson, Sanders, Seymour, Thomas (2000): Every 2-edge-connected cubic graph without a subgraph contractible to the Petersen graph has a 4-NZF.

- C. Q. Zhang and HJL (1992, DM): Every $4\log_2(|V(G)|)$-edge-connected graph has a 3-NZF.

- Y. Shao, H. Wu, J. Zhou and HJL (2008, JCT(B)): Every $3\log_2(|V(G)|)$-edge-connected graph has a 3-NZF.

- Z. H. Chen, H. Y. Lai and HJL (2002, DM): Tutte’s flow conjectures are valid if and only if they are valid within line graphs.
The Nonhomogeneous Case

- Given an orientation $D(G)$ with incidence matrix $D$, $G$ has an $A$-NZF $\iff Df = 0$ has a nowhere zero solution $f \in F^*(G, A)$. 

Any $b : V \not\in A$ with $\sum_{v \in V(G)} b(v) = 0$ is an $A$-zero-sum function. The set of all $A$-zero-sum functions is $Z(G, A)$. 

Group Connectivity of Graphs – p. 12/31
The Nonhomogeneous Case

- Given an orientation $D(G)$ with incidence matrix $D$, $G$ has an $A$-NZF $\iff Df = 0$ has a nowhere zero solution $f \in F^*(G, A)$.

- If $f \in F(G, A)$ and $b = \partial f$. Then

$$\sum_{v \in V(G)} b(v) = \sum_{v \in V(G)} \partial f(v) = 0.$$
The Nonhomogeneous Case

- Given an orientation $D(G)$ with incidence matrix $D$, $G$ has an $A$-NZF $\iff Df = 0$ has a nowhere zero solution $f \in F^*(G, A)$.

- If $f \in F(G, A)$ and $b = \partial f$. Then
  \[
  \sum_{v \in V(G)} b(v) = \sum_{v \in V(G)} \partial f(v) = 0.
  \]

- Any $b : V \mapsto A$ with $\sum_{v \in V(G)} b(v) = 0$ is an $A$-zero-sum function. The set of all $A$-zero-sum functions is $Z(G, A)$. 
Group connectivity of a graph

For a function $b \in Z(G, A)$, a function $f \in F^*(G, A)$ satisfying $Df = b$ is an $(A, b)$-NZF of $G$. 
For a function $b \in Z(G, A)$, a function $f \in F^*(G, A)$ satisfying $Df = b$ is an $(A, b)$-NZF of $G$.

If $\forall b \in Z(G, A)$, $G$ has an $(A, b)$-NZF, then $G$ is $A$-connected.
For a function $b \in Z(G, A)$, a function $f \in F^*(G, A)$ satisfying $Df = b$ is an $(A, b)$-NZF of $G$.

If $\forall b \in Z(G, A)$, $G$ has an $(A, b)$-NZF, then $G$ is $A$-connected.

For a 2-edge-connected graph $G$, $\Lambda_g(G) = \min\{k : G$ is $A$-connected, for every abelian group $A$ with $|A| \geq k \}$. 
For a function \( b \in Z(G, A) \), a function \( f \in F^*(G, A) \) satisfying \( Df = b \) is an \((A, b)\)-NZF of \( G \).

If \( \forall b \in Z(G, A) \), \( G \) has an \((A, b)\)-NZF, then \( G \) is \( A \)-connected.

For a 2-edge-connected graph \( G \), \( \Lambda_g(G) = \min \{ k : G \text{ is } A\text{-connected, for every abelian group } A \text{ with } |A| \geq k \} \).

\( \Lambda(G) \leq \Lambda_g(G) \).
New Results

Jeager et al (1992): If $G$ is a 3-edge-connected graph, then $\Lambda_g(G) \leq 6$. 

Jeager et al (1992) and HJL (1998): For the $n$-cycle $C_n$, $g(C_n) = n + 1$. 

New Results

- Jeager et al (1992): If $G$ is a 3-edge-connected graph, then $\Lambda_g(G) \leq 6$.
- Jeager et al (1992): If $G$ is a 4-edge-connected graph, then $\Lambda_g(G) \leq 4$. 
New Results

- Jeager et al (1992): If $G$ is a 3-edge-connected graph, then $\Lambda(g(G)) \leq 6$.

- Jeager et al (1992): If $G$ is a 4-edge-connected graph, then $\Lambda(g(G)) \leq 4$.

- Jeager et al (1992) and HJL (1998): For the $n$-cycle $C_n$, $\Lambda_g(C_n) = n + 1$. 
New Conjectures (JCT(B), 1992)

- Jeager et al (1992): If $G$ is a 3-edge-connected graph, then $\Lambda_g(G) \leq 5$. 
New Conjectures (JCT(B),1992)

- Jeager et al (1992): If $G$ is a 3-edge-connected graph, then $\Lambda_g(G) \leq 5$.

- Jeager et al (1992): If $G$ is a 5-edge-connected graph, then $\Lambda_g(G) \leq 3$. 
New Conjectures (JCT(B), 1992)

- Jeager et al (1992): If $G$ is a 3-edge-connected graph, then $A_g(G) \leq 5$.

- Jeager et al (1992): If $G$ is a 5-edge-connected graph, then $A_g(G) \leq 3$.

- Jeager et al (1992): There exists an integer $k > 0$ such that if $G$ is a $k$-edge-connected graph, then $A_g(G) \leq 3$. 
Planar Graphs

- X. Zhang and HJL, (2000, GC): If $G$ is a 3-edge-connected planar graph, then $\Delta_g(G) \leq 5$.

- Kral, Pangrac and Voss, (2006, JGT): There exists a family of 3-edge-connected planar graphs $G$ with $\Delta_g(G) = 5$.

- X. Li and HJL, (2006, JGT): If $G$ is a 5-edge-connected planar graph, then $\Delta_g(G) \leq 3$.

- Kral, Pangrac and Voss, (2006, JGT): There exists a family of 4-edge-connected planar graphs $G$ with $\Delta_g(G) = 4$. 
Planar Graphs

- X. Zhang and HJL, (2000, GC): If $G$ is a 3-edge-connected planar graph, then $\Lambda_g(G) \leq 5$.

- Kral, Pangrac and Voss, (2006, JGT): There exists a family of 3-edge-connected planar graphs $G$ with $\Lambda_g(G) = 5$. 
Planar Graphs

- X. Zhang and HJL, (2000, GC): If $G$ is a 3-edge-connected planar graph, then $\Lambda_g(G) \leq 5$.

- Kral, Pangrac and Voss, (2006, JGT): There exists a family of 3-edge-connected planar graphs $G$ with $\Lambda_g(G) = 5$.

- X. Li and HJL, (2006, JGT): If $G$ is a 5-edge-connected planar graph, then $\Lambda_g(G) \leq 3$. 
Planar Graphs

- X. Zhang and HJL, (2000, GC): If $G$ is a 3-edge-connected planar graph, then $\Lambda_g(G) \leq 5$.

- Kral, Pangrac and Voss, (2006, JGT): There exists a family of 3-edge-connected planar graphs $G$ with $\Lambda_g(G) = 5$.

- X. Li and HJL, (2006, JGT): If $G$ is a 5-edge-connected planar graph, then $\Lambda_g(G) \leq 3$.

- Kral, Pangrac and Voss, (2006, JGT): There exists a family of 4-edge-connected planar graphs $G$ with $\Lambda_g(G) = 4$. 
The line graph of $G$ is $L(G)$, with $V(L(G)) = E(G)$, where two vertices are adjacent in $L(G)$ iff corresponding edges are adjacent in $G$. 

Z. Chen, H. Y. Lai and H. J. L. (2002, DM): Tutte's 3-flow conjecture holds if and only if every 4-edge-connected line graph has a 3-NZF.

Y. Shao and H. J. L. (2008, EJC): If $G$ is a 4-edge-connected graph, then $g(L(G)) = 3$.

Y. Shao, H. Wu, J. Zhou and H. J. L. (2008, JCT(B)): If $G$ is $3\log_2(|V(G)|) - 1$-edge-connected, then $g(L(G)) = 3$. 

Group Connectivity of Graphs – p. 17/31
The line graph of $G$ is $L(G)$, with $V(L(G)) = E(G)$, where two vertices are adjacent in $L(G)$ iff corresponding edges are adjacent in $G$.

Z. Chen, H. Y. Lai and HJL (2002, DM): Tutte’s 3-flow conjecture holds if and only if every 4-edge-connected line graph has a 3-NZF.
The line graph of $G$ is $L(G)$, with $V(L(G)) = E(G)$, where two vertices are adjacent in $L(G)$ iff corresponding edges are adjacent in $G$.

Z. Chen, H. Y. Lai and HJL (2002, DM): Tutte’s 3-flow conjecture holds if and only if every 4-edge-connected line graph has a 3-NZF.

Y. Shao and HJL (2008, EJC): If $G$ is a 4-edge-connected graph, then $\Lambda_g(L(G)) \leq 3$. 
The line graph of $G$ is $L(G)$, with $V(L(G)) = E(G)$, where two vertices are adjacent in $L(G)$ iff corresponding edges are adjacent in $G$.

Z. Chen, H. Y. Lai and HJL (2002, DM): Tutte’s 3-flow conjecture holds if and only if every 4-edge-connected line graph has a 3-NZF.

Y. Shao and HJL (2008, EJC): If $G$ is a 4-edge-connected graph, then $\Lambda_g(L(G)) \leq 3$.

Y. Shao, H. Wu, J. Zhou and HJL (2008, JCT(B)): If $G$ is $3\log_2(|V(G)|)$-edge-connected, then $\Lambda_g(L(G)) \leq 3$. 
Complete Bipartite Graphs

J. Chen, E. Eschen and HJL (2008, Ars Comb): Let \( m \geq n \geq 2 \) be integers. Then

\[
\Lambda_g(K_{m,n}) = \begin{cases} 
5 & \text{if } n = 2 \\
4 & \text{if } n = 3 \\
3 & \text{if } n \geq 4
\end{cases}
\]

Let \( G \) be a graph with \( u'v' \in E(G) \) and \( H \) be a graph with \( uv \in E(H) \). \( G \oplus H \) denotes the graph obtained from the disjoint union of \( G - \{u'v'\} \) and \( H \) by identifying \( u' \) and \( u \) and identifying \( v' \) and \( v \).
Chordal Graphs:

A graph $G$ is **chordal** if every induced cycle $C$ of length at least 4 has a chord, an edge $e \in E(G) - E(C)$ both of whose ends are on $V(C)$.
Chordal Graphs:

- A graph $G$ is **chordal** if every induced cycle $C$ of length at least 4 has a chord, an edge $e \in E(G) - E(C)$ both of whose ends are on $V(C)$.

- As examples, all complete graphs or order at least 3 are chordal.
A graph $G$ is chordal if every induced cycle $C$ of length at least 4 has a chord, an edge $e \in E(G) - E(C)$ both of whose ends are on $V(C)$.

As examples, all complete graphs of order at least 3 are chordal.

**Problem:** Determine the group connectivity of chordal graphs.
If $G$ is a connected chordal graph, then $\Lambda_g(G) \leq 4$. 

Let $G$ be a 3-connected chordal graph. Then $g(G) = 3$ if and only if $G = K_4$.

Let $G$ be 2-connected (but not 3-connected) chordal graph. Then $g(G) = 4$ if and only if $G \not\cong K_3; K_4$ or $G$ has two subgraphs $G_1$ and $G_2$ such that both $g(G_1)$ and $g(G_2)$ are 4, and such that $G = G_1 \cup G_2$. 

Chordal Graphs: (J. Chen, E. Eschen and HJL)
Chordal Graphs: (J. Chen, E. Eschen and HJL)

- If $G$ is a connected chordal graph, then $\Lambda_g(G) \leq 4$.
- If $G$ is a 4-edge-connected chordal graph, then $\Lambda_g(G) \leq 3$. 
Chordal Graphs: (J. Chen, E. Eschen and HJL)

- If $G$ is a connected chordal graph, then $\Lambda_g(G) \leq 4$.
- If $G$ is a 4-edge-connected chordal graph, then $\Lambda_g(G) \leq 3$.
- Let $G$ be a 3-connected chordal graph. Then $\Lambda_g(G) = 3$ if and only if $G \not\cong K_4$. 

Group Connectivity of Graphs – p. 20/31
If $G$ is a connected chordal graph, then $\Lambda_g(G) \leq 4$.

If $G$ is a 4-edge-connected chordal graph, then $\Lambda_g(G) \leq 3$.

Let $G$ be a 3-connected chordal graph. Then $\Lambda_g(G) = 3$ if and only if $G \not\cong K_4$.

Let $G$ be 2-connected (but not 3-connected) chordal graph. Then $\Lambda_g(G) = 4$ if and only if $G \in \{K_3, K_4\}$ or $G$ has two subgraphs $G_1$ and $G_2$ such that both $\Lambda_g(G_1)$ and $\Lambda_g(G_2)$ are 4, and such that $G = G_1 \oplus G_2$. 
Graphs with Diameter at most 2

H.-J. Lai (1992, JGT): If $G$ is a 2-edge-connected graph with diameter at most 2, then $\Lambda(G) \leq 5$, where equality holds if and only if $G = P_{10}$, the Petersen graph.
Graphs with Diameter at most 2

- H.-J. Lai (1992, JGT): If $G$ is a 2-edge-connected graph with diameter at most 2, then $\Lambda(G) \leq 5$, where equality holds if and only if $G = P_{10}$, the Petersen graph.

- X. Yao and HJL (2006, EJC): If $G$ is a 2-edge-connected graph with diameter at most 2, then
  (i) $\Lambda(G) \leq 6$, and $\Lambda_g(G) = 6$ if and only if $G = C_5$.
  (ii) If $G \neq C_5$, then $\Lambda_g(G) \leq 5$, where equality holds if and only if $G = P_{10}$, the Petersen graph, or $G \in \{S_{m,n}, K_{2,n}\}$. 
Graphs with Diameter at most 2

$K_{2,m}$

$S_{m,n}$
Triangulated Graphs

A $G$ is triangulated if every edge of $G$ lies in a cycle of length at most 3 in $G$. 

Conjecture (Xu and Zhang, 2002)

If $G$ is a 4-edge-connected triangulated graph, then \( (G) \geq 3 \).

Conjecture (Davos, 2003)

If $G$ is a 4-edge-connected triangulated graph, then \( g(G) \geq 3 \).
Triangulated Graphs

- A graph $G$ is triangulated if every edge of $G$ lies in a cycle of length at most 3 in $G$.

- Conjecture (Xu and Zhang, 2002) If $G$ is a 4-edge-connected triangulated graph, then $\Lambda(G) \leq 3$. 

Conjecture (Davos, 2003)
A graph $G$ is triangulated if every edge of $G$ lies in a cycle of length at most 3 in $G$.

Conjecture (Xu and Zhang, 2002) If $G$ is a 4-edge-connected triangulated graph, then $\Lambda(G) \leq 3$.

Conjecture (Davos, 2003) If $G$ is a 4-edge-connected triangulated graph, then $\Lambda_g(G) \leq 3$.
Xu, Zhou and HJL (2008 GC) found an infinite family of 4-edge-connected triangulated graphs $G$ with $\Lambda_g(G) = 4$. 
Triangulated Graphs

A $G$ is **triangularly connected** if every pair of edges of $G$ are joined by a sequence of consecutively intersecting 3-cycles in $G$. 

Theorem (Fan, Xu, Zhang, Zhou and HJL, 2008, JCT(B))

If $G$ is a triangularly-connected graph, then $g(G) \leq 3$ iff $G$ cannot be obtained by a sequence of parallel connections from fans and/or odd wheels.

Corollary

If $G$ is a 3-edge-connected, triangularly-connected graph, then $g(G) \leq 3$. 

Group Connectivity of Graphs – p. 25/31
Triangulated Graphs

A graph $G$ is **triangularly connected** if every pair of edges of $G$ are joined by a sequence of consecutively intersecting 3-cycles in $G$.

**Theorem** (Fan, Xu, Zhang, Zhou and HJL, 2008, JCT(B)) If $G$ is a triangularly-connected graph, then $\Lambda_g(G) \leq 3$ iff $G$ cannot be obtained by a sequence of parallel connections from fans and/or odd wheels.

**Corollary** If $G$ is a 3-edge-connected, triangularly-connected graph, then $\Lambda_g(G) \leq 3$. 
A $G$ is **triangularly connected** if every pair of edges of $G$ are joined by a sequence of consecutively intersecting 3-cycles in $G$.

**Theorem** (Fan, Xu, Zhang, Zhou and HJL, 2008, JCT(B)) If $G$ is a triangularly-connected graph, then $\Lambda_g(G) \leq 3$ iff $G$ cannot be obtained by a sequence of parallel connections from fans and/or odd wheels.

**Corollary** If $G$ is a 3-edge-connected, triangularly-connected graph, then $\Lambda_g(G) \leq 3$. 

**Triangulated Graphs**

Group Connectivity of Graphs – p. 25/31
Barat-Thomassen’s Approach

- A graph $G$ with $|E(G)| \equiv 0 \pmod{3}$ has a **claw-decomposition** if $E(G)$ is a disjoint union $E(G) = X_1 \cup X_2 \cup \cdots \cup X_k$ such that for each $i$ with $1 \leq i \leq k$, $G[X_i]$ is a generalized claw.

*Theorem (Barat and Thomassen, 2006, JGT)*

There exists a function $f(k)$ such that if every $k$-edge-connected graph $G$ with $|E(G)| \equiv 0 \pmod{3}$ has a claw-decomposition, then every $f(k)$-edge-connected graph $G$ has a 3-NZF.
Barat-Thomassen’s Approach

A graph $G$ with $|E(G)| \equiv 0 \pmod{3}$ has a **claw-decomposition** if $E(G)$ is a disjoint union $E(G) = X_1 \cup X_2 \cup \cdots \cup X_k$ such that for each $i$ with $1 \leq i \leq k$, $G[X_i]$ is a generalized claw.

**Theorem (Barat and Thomassen, 2006, JGT)** There exists a function $f(k)$ such that if every $k$-edge-connected graph $G$ with $|E(G)| \equiv 0 \pmod{3}$ has a claw-decomposition, then every $f(k)$-edge-connected graph $G$ has a 3-NZF.
Barat-Thomassen’s Approach

- Conjecture (Barat and Thomassen, 2006, JGT) Every 4-edge-connected simple planar graph $G$ with $|E(G)| \equiv 0 \pmod{3}$ has a claw-decomposition.
Counterexample

(HJL, SIAM J of DM, 2007) An infinite family of 4-edge-connected simple planar graph $G$ with $|E(G)| \equiv 0 \pmod{3}$ is constructed which does not have a $K_{1,3}$-decomposition.
Suppose $G$ has a claw-decomposition $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$, and let $D = D(\mathcal{X})$ (All edges oriented towards the center of the claw).
Suppose $G$ has a claw-decomposition $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$, and let $D = D(\mathcal{X})$ (All edges oriented towards the center of the claw).

$\forall v \in V(G)$, $|E^+_D(v)| \in \{0, 3\}$. As $|V(G)| = 24k$ and $|E(G)| = 48k$, so $G$ has $m = 48k/3 = 16k$ edge-disjoint claws.
Suppose $G$ has a claw-decomposition $\mathcal{X} = \{X_1, X_2, \cdots, X_m\}$, and let $D = D(\mathcal{X})$ (All edges oriented towards the center of the claw).

$\forall v \in V(G), |E_D^+(v)| \in \{0, 3\}$. As $|V(G)| = 24k$ and $|E(G)| = 48k$, so $G$ has $m = 48k/3 = 16k$ edge-disjoint claws.

Let $H_i$ ($i = 1, 2, \ldots, 3k$) denote a "building block".
Let $W = \{v \text{ with } |E_D^+(v)| = 0\}$. Then

$$|W| = |V(G)| - m = 24k - 16k = 8k.$$
Let \( W = \{ v \text{ with } |E_D^+(v)| = 0 \} \). Then
\[
|W| = |V(G)| - m = 24k - 16k = 8k.
\]

No two vertices in \( W \) can be adjacent in \( G \), and so for each \( i \) (mod \( 3k \)), \(|W \cap V(H_i \cup H_{i+1} - \{y_{i+1}\})| \leq 5\).
Let $W = \{ v \text{ with } |E_D^+(v)| = 0 \}$. Then

$$|W| = |V(G)| - m = 24k - 16k = 8k.$$ 

No two vertices in $W$ can be adjacent in $G$, and so for each $i \pmod{3k}$, $|W \cap V(H_i \cup H_{i+1} - \{y_{i+1}\})| \leq 5$.

$$16k = 2|W| = \sum_{i=1}^{3k} |V(H_i \cup H_{i+1} - \{y_{i+1}\}) \cap W| \leq 5 \times 3k = 15k,$$ a contradiction.
Thank You!