Part III

SUGEGRAPHS
7. Subgraphs of graphs, II

In [6] we gave a sufficient condition for $H$ to be a subgraph of $G$ by showing that for any positive integer $d$ there is a constant $c_d < d$ such that $\Delta(H) \leq d$ and $\delta(G) \geq c_d p$ imply that $H$ is a subgraph of $G$. We obtained this from a result on bipartite graphs that is analogous to Theorem 7.1, and is in a certain sense best possible. In a footnote in [6] we announced having improved $c_d$ to the value given by Theorem 7.1 below. Like Theorem 7.1, our result in [6] on bipartite graphs may be obtained using a generalization of the concept of alternating paths, which is used extensively in studying matchings. In the special case when $\Delta(H) = 1$, the proof of Theorem 7.1 reduces to an argument involving an alternating path of length 4.

N. Sauer and J. Spencer [14] have independently obtained Theorem 7.1. This was announced in [13]. Erdős and Stone [9] gave a sufficient condition of a different nature for $H$ to be a subgraph of $G$. Bollobás and Eldridge [3] and Sauer and Spencer [14] have considered
the problem of giving sufficient conditions, based on the number of edges of $H$ and $G^c$, for $H$ to be a subgraph of $G$.

After proving the main result of section 7, we give some special cases, and indicate what would be best possible. Our main result of this section is

**Theorem 7.1** If $G$ and $H$ are graphs on $p$ vertices such that

$$(7.1) \quad 2\Delta(G^c)\Delta(H) \leq p - 1$$

then $H$ is a subgraph of $G$.

**Proof:** Throughout the proof, the letter $w$ will be used to denote vertices of $H$ (i.e., $w \in V(H)$), and the letters $x$ and $v$ will be used for vertices of $G$. Given a graph $G$, suppose that $H$ is an edge-minimal graph that is not a subgraph of $G$, but suppose that $H$ and $G$ satisfy (7.1). By edge-minimality, we can pick any edge, say $e \in V(H)$, fixed throughout the proof, so that $H - e$ is a subgraph of $G$. Let

$$\pi: V(H) \longrightarrow V(G)$$

be an embedding of $H - e$ into $G$. Let $w, w'$ be the ends of $e$ in $H$. We shall alter $\pi$ by transposing $\pi(w)$ with another vertex of $G$ so that the resulting embedding of $H - e$ also maps $e$ to an edge of $G$. This is a contradiction.
Define
\[ M(v) = \{ v'' \in V(G) : \pi^{-1}(v), \pi^{-1}(v'') \in E(H - e) \} \].

A successor of \( v \) is any vertex \( v_1 \in V(G) \) such that for each \( v'' \in M(v) \), \( v_1 \) is either equal or adjacent in \( G \) to \( v'' \). Denote by \( S(v) \) the set of all successors of \( v \). We say that \( v \) is a predecessor of \( v_1 \) if \( v_1 \) is a successor of \( v \). Denote by \( P(v_1) \) the set of all predecessors of \( v_1 \).

Let \( v = \pi(w) \). Note that if \( v_1 \in S(v) \cap P(v) \) and if \( v_1 \neq v \), then \((v, v_1)\) embeds \( H - e \) into \( G \).

We estimate \(|S(v)|\) and \(|P(v)|\) by deriving upper bounds for \(|V(G) - S(v)|\) and \(|V(G) - P(v)|\). A vertex \( x \in V(G) \) is outside \( S(v) \) if \( x \) is adjacent in \( G^c \) to a vertex \( x' \) of \( M(v) \). For any given \( x' \in M(v) \), there are \( \Delta(G^c) \) choices of \( x \) adjacent to \( x' \) in \( G^c \). Since \( \deg_{H - e}(w) \leq \Delta(H) - 1 \), we must have \( |M(v)| \leq \Delta(H) - 1 \) choices of \( x' \). Hence, at most \( \Delta(G^c)(\Delta(H) - 1) \) vertices \( x \) are not in \( S(v) \). If \( x \notin P(v) \), then there is an \( x' \in M(x) \) such that \( x' \) is adjacent in \( G^c \) to \( v \). There are at most \( \Delta(G^c) \) choices of \( x' \) adjacent to \( v \) in \( G^c \), and one of them is \( \pi(w') \), since \( \pi \) does not embed \( e \) into \( G \). Each \( x' \) lies in at most \( \Delta(H) \) sets \( M(x) \), where \( x \in V(G) \), with strict inequality when \( \pi^{-1}(x') = w' \), whence at most \( \Delta(G^c) \Delta(H) - 1 \) vertices \( x \) of \( V(G) \) are not in \( P(v) \).
Therefore,
\[ |P(v) \cap S(v)| \geq |V(G)| - |V(G) - P(v)| - |V(G) - S(v)| \]
\[ \geq p - (\Delta(G^c) \Delta(H) - 1) \]
\[ - \Delta(G^c)(\Delta(H) - 1) \]
\[ = p - 2 \Delta(H) \Delta(G^c) + \Delta(G^c) + 1 \]
\[ \geq 2 + \Delta(G^c), \]
by (7.1). At most \(1 + \Delta(G^c)\) vertices are not adjacent in \(G\) to \(\pi(w')\). Therefore, there is a \(v_1 \in P(v) \cap S(v)\) that is adjacent to \(\pi(w')\) in \(G\). Thus, \((v,v_1)\pi\) is an embedding of \(H\) into \(G\). This proves the theorem.

**Conjecture** If \(G\) and \(H\) are graphs on \(p\) vertices satisfying
\[ (\Delta(H) + 1)(\Delta(G^c) + 1) \leq p + 1, \]
then \(H\) is a subgraph of \(G\).

We give examples to show that the conjecture, if true, would be best possible. Let \(H\) be a graph on \(p\) vertices, and let \(d\) be an integer such that
\[ p \equiv -2 \pmod{d+1}. \]
Then \(H\) is said to be in class \(C_1(d)\) if \(\Delta(H) = d\) and if \(H\) has \(\frac{p+2}{d+1} - 1\) components isomorphic to \(K_{d+1}\); \(H\) is in class \(C_2(d)\) if \(d\) is odd, if \(H\) has one component isomorphic to \(K_{d,d'}\), and if all \(\frac{p+2}{d+1} - 2\) other components are isomorphic to \(K_{d+1}\). Thus, for any odd \(d\), there is a unique graph in \(C_2(d)\), and for \(d\) even, \(C_2(d)\) is empty. We also denote these classes by \(C_1^e\) and \(C_2^e\) (Figures 1, 2, 3).
Figure 1: The graph in $C_1(2)$ with $p=10$.

Figure 2: A graph in $C_1(3)$ with $p=10$.

Figure 3: The graph in $C_2(3)$ with $p=10$. 
For any integers \(d, d'\) satisfying
\[
(d + 1)(d' + 1) = p + 2,
\]
if \(H \subseteq C_1(d) \cup C_2(d)\) and if \(G^c \subseteq C_1(d') \cup C_2(d')\), then it is easily verified that \(H\) is not a subgraph of \(G\), unless \(H \subseteq C_2(d)\) and \(G^c \subseteq C_2(d')\). Thus, the conjecture, if true, is best possible.

It is routine to show that if (7.1) of Theorem 7.1 is improved to
\[
2\Delta(H)\Delta(G^c) \leq p,
\]
then either \(H\) is a subgraph of \(G\), or one of \(H\) or \(G^c\), say \(H\), is regular of degree 1, and the other graph lies in \(C_1(p/2)\) or \(C_2(p/2)\). The proof, however, becomes much longer than the proof of Theorem 7.1.

We have recently been able to show that there is a function \(f(p)\), on the order of \(p^{1/3}\), such that if \(G\) and \(H\) are graphs on \(p\) vertices with \(\Delta(H) = 2\) and \(\Delta(G^c) \leq \frac{p}{3} - f(p)\), then \(H\) is a subgraph of \(G\). The coefficient \(\frac{1}{3}\) is best possible by above examples. A proof is in section 10. In the special case where \(H\) has \(\left\lfloor \frac{p}{3} \right\rfloor\) components isomorphic to \(K_3\) and \(G\) satisfies
\[
\Delta(G^c) \leq \frac{p-1}{3},
\]
either \(H\) is a subgraph of \(G\), or equality holds and \(G^c \subseteq C_1\left(\frac{p-1}{3}\right) \cup C_2\left(\frac{p-1}{3}\right)\). This characterizes the extremal graphs of a theorem of Corrádi and Hajnal [7]. We prove this in section 9.
There is a more general special case for which the conjecture has been proved. If
\[ b = \frac{p+1}{\Delta(H)+1} - 1 \]
is an integer, then this value of \( b \) in the following theorem of Hajnal and Szemerédi [10] gives the inequality of the conjecture. The case \( \Delta(H) = 2 \) is the theorem of Corrádi and Hajnal.

**Theorem 7.2** Let \( H \) be a graph with \( b \) components isomorphic to \( K_{\Delta(H)+1} \), and with all other components isomorphic to \( K_{\Delta(H)} \). If
\[ \Delta(G^c) \leq \frac{p-b}{\Delta(H)} - 1, \]
then \( H \) is a subgraph of \( G \).

Theorem 7.2 can be readily derived from the special case with \( b = \frac{p+1}{\Delta(H)+1} - 1 \). In this case \( H \in C_1 \). Hajnal and Szemerédi gave \( G^c \in C_1 \) to show that their result is best possible. The example \( G^c \in C_2 \) is new.

Given a graph \( H \), the conjecture is not necessarily best possible for that particular graph. For instance, consider the following theorem of Bondy [4]:

**Theorem 7.3** If \( H \) is a graph consisting of one polygonal component of girth \( g \leq p \) and of \( p-g \) components \( K_1 \), and if \( G \) satisfies \( \Delta(G^c) \leq \frac{p}{3} - 1 \) and also has \( p \) vertices, then either \( H \) is a subgraph of \( G \), or \( g \) is odd, \( p \) is even and \( G \) is isomorphic to \( K_{\frac{p}{3}}, \frac{p}{3} \).
Let \( n = \binom{p}{2} \). If \( G^c \) contains a component isomorphic to \( K_{n+1} \) and \( g = p \), or if \( G^c \) has a component isomorphic to \( K_{n,n} \) and \( g > p - n \), then \( H \), of Theorem 7.3, is not a subgraph of \( G \). These examples are similar to the classes \( C_1 \) and \( C_2 \) which make Theorem 7.1 best possible for \( \Delta(H) = 1 \), and which make the conjecture best possible.

Next, we give a general construction of examples of graphs \( G \) such that a given graph \( H \) is not a subgraph of \( G \). This construction generalizes class \( C_1 \) given above.

Let \( r(H) \) denote the minimum number of vertices whose removal from \( H \) is necessary to lower the chromatic number \( \chi(H) \). Clearly,

\[
r(H) \leq \frac{p}{\chi(H)}.
\]

**Theorem 7.4** Let \( H \) be a graph on \( p \) vertices. There is a graph \( G \) on \( p \) vertices with

\[
\Delta(G^c) = \left\lfloor \frac{p - r(H)}{\chi(H) - 1} \right\rfloor
\]

such that \( H \) is not a subgraph of \( G \).

**Proof:** Let \( n = \chi(H) - 1 \). Partition the \( p \) vertices of a set \( X \) into sets \( X_0, X_1, \ldots, X_n \), where

\[
|X_0| = r(H) - 1,
\]

where \( X_0 \) is empty if \( r(H) = 1 \), where

\[
|X_1| \leq |X_2| \leq \cdots \leq |X_n|,
\]

and \( |X_n| - |X_1| \leq 1 \). Let a graph \( G \) be defined on \( X \) so that \( G \) is the complete \((n+1)\)-partite graph with \((n+1)\)-partition \( X_0, X_1, \ldots, X_n \). Since
\[ |X_0| = r(H) - 1 \leq \frac{\Delta}{\chi(H)} - 1. \]

and since \( \frac{\Delta}{\chi(H)} \) is the average of \( |X_i|, i = 0, 1, \ldots, n \), we must have \( |X_n| > |X_0| \). Thus, \( G[C[X_n]] \) is a clique, and letting braces denote the least integer function, we have

\[
\Delta(G^c) = |X_n| - 1 \\
= \left\{ (p - |X_0|)/n \right\} - 1 \\
= \left[ (p - |X_0| + n - 1)/n \right] - 1 \\
= \left[ (p - |X_0| - 1)/n \right] \\
= \left[ (p - r(H))/(\chi(H) - 1) \right],
\]

which is the condition of the theorem.

Suppose that \( \pi \) embeds \( H \) into \( G \). Let \( H' \) denote the subgraph of \( H \) induced by the preimage of \( V(G) - X_0 \).

Since \( H' \) contains \( |X_0| = r(H) - 1 \) fewer vertices than \( H \), by the definition of \( r \),

\[ \chi(H') = \chi(H). \]

But for each \( i = 1, 2, \ldots, n \), no two vertices of \( X_i \) are adjacent in \( G \), and so the embedding

\[ \pi|_{H'}: V(H') \longrightarrow V(G) - X_0 \]

is a \((\chi(H) - 1)\)-coloring of \( H' \), a contradiction. Hence, \( \pi \) does not exist, and the theorem is proved.

A \( B_h \)-component is defined in section 2.

**Corollary 2.5** Let \( H \) be a graph on \( p \) vertices of maximum degree \( \Delta(H) \) with \( b > 0 \) \( B_h \)-components. Then there is a graph \( G \) with \( \Delta(G^c) = \lfloor (p - b)/\Delta(H) \rfloor \), such that \( H \) is a subgraph of \( G \).
Proof: By Theorem 2.1, due to Brooks, we have
\[ \chi(H) = \Delta(H) + 1, \quad r(H) = b \text{ in Theorem 7.4.} \]

Note that Theorem 7.4 and Corollary 7.5 contain, as special cases, some of the aforementioned examples showing that certain results are best possible.

Consider Bondy's Theorem. If \( p \) and \( g \) are odd, then \( b = 1 \) and Corollary 7.5 shows Theorem 7.3 to be best possible. If \( g \) is even and equal to \( p \), then \( r(H) = \frac{g}{2p} \), and Theorem 7.4 shows that \( \Delta(G^c) \leq \frac{g}{2p} \) is not sufficient to ensure that \( H \) is a subgraph of \( G \).

In the case of Hajnal and Szemeredi's Theorem, Corollary 7.5 ensures that it is best possible for any value of \( \Delta(H) \) and any \( b \geq 1 \).

Let \( H \) be a graph satisfying the conditions of Corollary 7.5. We know of no graph on \( p \) vertices with
\[ \Delta(G^c) \leq \frac{p - b}{\Delta(H)} - 1, \]
even when \( b = 0 \), such that \( H \) is not a subgraph of \( G \). It would be interesting to know whether such graphs exist.

We give in this section an improvement of Theorem 7.1 for the case $\Delta(H) = 2$ that has a short proof, but is not best possible. In section 10, we prove a stronger result, which is best possible in a certain sense.

As in section 7, the graph $H$ will be embedded in $G$, and the letters $y$ and $w$ will be used to denote the vertices of $H$, while $x$ and $v$ will be used for vertices of $G$. In this section, let

$$M(v) = \{v'' \in V(G): \{\pi^{-1}(v), \pi^{-1}(v'')\} \in E(H)\},$$

where

$$\pi: V(H) \rightarrow V(G)$$

is a fixed bijection.

In this section we used a slightly different definition of successors and predecessors. A vertex $x' \in V(G)$ is a **successor** of $x \in V(G)$ if $x'$ is adjacent in $G$ to every vertex of $M(x)$. (In section 7, we permitted $x'$ to be a successor of $x$ if $x'$ were adjacent or equal to every vertex of $M(x)$.) The vertex $x$ is a **predecessor** of $x'$ if $x'$ is a successor of $x$. 

56
Theorem 8.1 Let $G$ and $H$ be graphs on $p$ vertices, with $\Delta(H) = 2$. If
\[
\Delta(G^c) < \frac{2p - 11}{7},
\]
then $H$ is a subgraph of $G$.

Proof: Let $H$ be an edge-minimal graph for which the theorem is false. Then there is a graph $G$ satisfying (8.1) such that $H$ is not a subgraph of $G$, but such that for any edge $e \in E(H)$, $H - e$ is a subgraph of $G$.

First, we show that if any vertex $y_1 \in V(H)$ has degree 1 in $H$, then we are done. Let $e = \{y_0, y_1\}$ be the edge incident with $y_1$, and let
\[
\pi: V(H) \rightarrow V(G)
\]
be an embedding of $H - e$ into $G$. Since
\[
\Delta(G^c) < \frac{2p}{7},
\]
$\pi(y_1)$ has at least $\frac{2p}{7}$ predecessors in $G$, and (8.1) guarantees that among them lie successors of $\pi(y_1)$ (i.e., vertices of $G$ adjacent to $\pi(y_0)$). Let $x$ be such a vertex. Then $(x \pi(y_1))\pi$ is an embedding of $H$ into $G$.

Therefore, assume that each vertex of $H$ has degree either 0 or 2. Thus, all components of $H$ are either isolated vertices or polygons.

Let the edges of polygons of $H$ be directed so that each vertex has one incoming edge and one outgoing edge.
Given a vertex $y_0$ in $H$, we shall denote by $y_1, y_2, y_3$ the next three successive vertices on the directed path in $H$ starting at $y_0$. On a triangle, $y_0 = y_3$.

For the mapping 

$$\pi: V(H) \rightarrow V(G),$$

we shall simplify notation by writing $\pi(y_i) = x_i$ for $i = 0, 1, 2, 3$.

By the minimality of $H$, we may assume that $\pi$ embeds all but one edge, say $e$, of $H$ into $G$. Denote the tail of $e$ by $w_0$. Following the previous convention, the head of $e$ is $w_1$, and the next two successive vertices after $w_1$ are $w_2$ and $w_3$. To simplify notation, we write $\pi(w_i) = v_i$, $i = 0, 1, 2, 3$.

Since $e$ is the only unembedded edge of $H$, $x_0, x_1, x_2, x_3$ and $v_0, v_1, v_2, v_3$ are paths in $G$, and $v_0$ and $v_1$ are not adjacent in $G$.

Throughout the proof, we consider $e, w_0, \pi,$ and $v_0$ to be fixed. We shall choose $y_0$ so that the paths $y_0, y_1, y_2, y_3$ and $w_0, w_1, w_2, w_3$ have no edge in common. Hence, $w_3 \neq y_1, y_2,$ or $y_3$, and $w_0 \neq y_1$ or $y_2$ (the case $w_0 = y_0$ is excluded by $w_3 \neq y_3$). For any other choice of $y_0 \in V(H)$, the two paths have no common edge. Thus, we have $p - 5$ choices for $y_0$. Each choice determines $x_0, x_1, x_2,$ and $x_3$ since $\pi$ is fixed.
For any choice of $y_0$, if $(v_1 x_1)^\pi$ or $(v_1 x_1)(v_2 x_2)^\pi$
is an embedding, then $H$ is a subgraph of $G$, and we are
done. Otherwise, if neither is an embedding, then
$E(G^c)$ includes some of the following six edges:

$(8.2) \{v_0, x_1\}, \{v_1, x_0\}, \{v_1, x_2\}, \{v_2, x_1\}, \{v_2, x_3\}, \{v_3, x_2\}$.

We shall estimate the number of values of $y_0$ for
which $(v_2 x_2)^\pi$ embeds $H - e$ into $G$. Observe that for
a given $y_0$ (which determines $x_0, x_1, x_2, x_3$),

$(8.3)$ If exactly one of the six edges $(8.2)$ is
in $E(G^c)$, then it must be $\{v_0, x_1\}$ or $\{v_1, x_0\}$
(otherwise, $(v_1 x_1)^\pi$ or $(v_1 x_1)(v_2 x_2)^\pi$ embeds $H$
into $G$), and so $(v_2 x_2)^\pi$ embeds $H - e$ into $G$.

Let $n_1$ be the number of values of $y_0$ such that
$(8.3)$ holds. This leaves $p - 5 - n_1$ choices of $y_0$ such
that at least 2 of the 6 edges $(8.2)$ lie in $E(G^c)$.

We count occurrences of edges in $E(G^c)$ among the 6
edges of $(8.2)$ in two ways, as $y_0$ runs over $p - 5$ ver-
tices in $H$. It is clear that their number is at least

$$1(n_1) + 2(p - 5 - n_1) = 2p - 10 - n_1.$$ 

Also, each of at most $\Delta(G^c)$ edges incident with $v_1$ is
counted once among the 6 edges $(8.2)$, if $i = 0$ or 3,
and each is counted twice if $i = 1$ or 2. Hence, the
number of edge-occurrences is at most $6 \Delta(G^c)$. 
Counting two ways, we get
\[ 2p - 10 - n_1 \leq \text{number of edge occurrences} \leq 6 \Delta(G^c). \]

Hence,
\[ n_1 \geq 2p - 10 - 6 \Delta(G^c), \]
and so by (8.1),
\[ n_1 \geq \Delta(G^c) + 1. \]

Thus, there are at least \( \Delta(G^c) + 1 \) values of \( y_0 \), and hence \( \Delta(G^c) + 1 \) values of \( x_2 \), such that \((v_2, x_2)^n\) embeds \( H - e \) into \( G \).

The number of vertices \( x \) such that both vertices of \( \{x, y(x)\} \) are adjacent in \( G \) to \( v_1 \) (i.e., the number of predecessors of \( v_1 \)) is at least
\[ p - 2 \Delta(G^c) \geq \frac{3p + 22}{7}. \]

At most \((2p - 10)/7\) of these are not adjacent to \( v_0 \), and so the number of predecessors of \( v_1 \) that are adjacent to \( v_0 \) in \( G \) is at least \( p + 11 \). Let \( x \) be any one of these. Among the \( \Delta(G^c) + 1 \) values of \( x_2 \) such that \((v_2, x_2)^n\) embeds \( H - e \) into \( G \), choose \( x_2 \) to be adjacent to \( x \). Then \((v_1, x)(v_2, x_2)^n\) embeds \( H \) into \( G \). But this contradicts the assumption that \( H \) is not a subgraph of \( G \). The proof is complete.
9. Subgraphs with triangular components

In section 7 we gave two classes of graphs, denoted \( C_1(d) \) and \( C_2(d) \), such that if
\[
(d+1)(d'+1) = p+2,
\]
if \( H \in C_1(d') \cup C_2(d') \), and if \( G^c \in C_1(d) \setminus C_2(d) \), then either \( H \) is not a subgraph of \( G \) or both \( H \in C_2(d') \) and \( G^c \in C_2(d) \). We conjectured that if
\[
(\Delta(G^c)+1)(\Delta(H)+1) \leq p+1,
\]
then \( H \) is a subgraph of \( G \). Thus, these two classes \( C_1 \) and \( C_2 \) make the conjecture best possible.

To simplify notation in this section, we shall say that \( G \) is of type 1 or type 2 if \( p = 3b+1 \), \( b > 0 \), and either \( G^c \in C_1(b) \) or \( G^c \in C_2(b) \), respectively. Thus, when \( G \) is of type 1, there is a set \( S \) of \( b-1 \) vertices such that \( G - S \) is isomorphic to \( K_{b+1,b+1} \). Also, when \( G \) is of type 2, there is a stable set \( S \) of \( b+1 \) vertices such that \( G - S \) has 2 components, both isomorphic to \( K_b \), and \( b \) is odd.

Suppose \( H \in C_1(2) \). If \( G \) is of type 1 or type 2, then clearly \( H \) is not a subgraph of \( G \). We shall show that if \( H \in C_1(2) \), then graphs \( G \) of types 1 and 2 are
the only graphs with \( \delta(G) \geq \frac{p-1}{3} \) such that \( H \) is not a subgraph of \( G \).

**Theorem 9.1** Let \( G \) and \( H \) be graphs on \( p \) vertices, and suppose that every component of \( H \) is isomorphic to either \( K_1, K_2, \) or \( K_3 \). Let \( b = b(H) \) denote the number of triangular components of \( H \), and suppose \( b \geq 0 \). If
\[
\delta(G) \geq \left\lceil \frac{p+b}{2} \right\rceil,
\]
and if \( H \) is not a subgraph of \( G \), then either

1. (9.1) There is a set \( S \) of \( b-1 \) vertices of \( G \) such that \( G-S \) is a complete bipartite graph; or
2. (9.2) There is a set \( S \) of \( b+1 \) vertices, \( b \) odd, such that \( G-S \) has two components, both isomorphic to \( K_b \), and \( H \) has \( \frac{p-1}{3} \) triangles.

**Theorem 9.2** Let \( G \) and \( H \) be graphs on \( p \) vertices and suppose that every component of \( H \) is a triangle \( K_3 \), except for one vertex \( K_1 \) if \( p = 3b+1 \), or one edge \( K_2 \) if \( p = 3b+3 \). If
\[
\delta(G) \geq \frac{2}{3}(p-1),
\]
then \( H \) is not a subgraph of \( G \) if and only if both
\[
\delta(G) = \frac{2}{3}(p-1) = 2b
\]
and \( G \) is of type 1 or type 2.

If \( H \) is the graph of Figure 1, then Figures 2 and 3, respectively, are the complements of corresponding graphs of types 1 and 2 such that \( H \) is not a subgraph.
Lemma 9.3 Let $G$ be a graph with $p = 3b + 1$ vertices, for some integer $b$, and with $\delta(G) \geq 2b$. If for some set $S \subseteq V(G)$, with $|S| = b - 1$, $G - S$ is bipartite, with bipartition $V_1 \cup V_2$, then the following conclusions hold:

Every vertex of $S$ is adjacent to every vertex of $G - S$;

$|V_1| = |V_2|$;

$G - S$ is a complete bipartite graph.

Thus, $G$ is of type 1.

Proof: Without loss of generality, assume that $|V_1| \geq |V_2|$. We have

$|V_1| \geq \frac{1}{2}(p - |S|) = \frac{1}{2}(3b + 1 - (b - 1)) = b + 1$.

Let $v_1 \in V_1$. Since $V_1 \cup V_2$ is a bipartition of $G - S$, $v_1$ is adjacent in $G^c$ to every vertex of $V_1 - v_1$. But

$\Delta(G^c) = p - \delta(G) - 1 \leq b$,

and hence we must have $|V_1| = b + 1$ and $\delta(G) = 2b$. Also, each $v_1 \in V_1$ must be adjacent to every vertex of $G - V_1$ (i.e., to every vertex of $V_2$ and every vertex of $S$). The conclusions of the lemma follow directly.

Remarks: If $G$ is of type 2, then $p = 4 \pmod{6}$, and $G$ is regular of degree $2b = \frac{2}{3}(p - 1)$. Note that the only graph that is both of type 1 and type 2 is the quadrilateral.
Lemma 9.4 Let $G$ be a graph with $p = 3b + 1$ vertices, for some integer $b$, and with $\delta(G) \geq 2b$. If for some set $S \subseteq V(G)$, with $|S| = b + 1$, $G - S$ has two components, then the following conditions hold:

Every vertex of $S$ is adjacent to every vertex of $G - S$;

$G - S$ has two components, both isomorphic to $K_b$.

If, furthermore, $b$ pairwise disjoint triangles do not embed in $G$, then

$p \equiv 4 \pmod{6}$;

$S$ is a stable set;

$G$ is of type 1 only if $G$ is a quadrilateral.

Thus, $G$ is of type 2.

Proof: Let $G$ and $S$ satisfy the hypotheses. Since $p = 3b + 1$ and $\delta(G) \geq 2b$, any vertex is adjacent in $G^c$ to at most $b$ vertices of $G$. Thus, since $|V(G - S)| = 2b$ and since $G - S$ has two components, any vertex in the smaller component is adjacent in $G^c$ to at least $\frac{1}{3} |V(G - S)| = b$ vertices in the larger component of $G - S$. But these statements force equality: both components have just $b$ vertices. Also, the first two conclusions of the lemma follow immediately.

If $S$ is not a stable set or if $p \not\equiv 4 \pmod{6}$, then either $G[S]$ has an edge, or, since $p = 3b + 1$, $p \equiv 1 \pmod{6}$. In either case, an embedding of $b$ pairwise disjoint triangles is easily found. The rest is easy.
Proof of Theorem 9.1 from Theorem 9.2: Assume without loss of generality that the components of $H$ consist of $b$ triangles $K_3$, $\left\lfloor \frac{p - 3b}{2} \right\rfloor$ edges $K_2$, and $p - 3b - 2\left\lfloor \frac{p - 3b}{2} \right\rfloor$ vertices $K_1$. By adding $\left\lfloor \frac{p - 3b}{2} \right\rfloor$ vertices to $H$, each adjacent to both ends of a $K_2$, we can construct a graph $H'$ on $p + \left\lfloor \frac{p - 3b}{2} \right\rfloor$ vertices, where the components of $H'$ consist of $b + \left\lfloor \frac{p - 3b}{2} \right\rfloor$ triangles $K_3$ and $p - 3b - 2\left\lfloor \frac{p - 3b}{2} \right\rfloor (= 0 \text{ or } 1)$ vertices $K_1$. By adding a stable set of $\left\lfloor \frac{p - 3b}{2} \right\rfloor$ vertices to $G$, we construct a graph $G'$ in which each added vertex is adjacent to every vertex of $G$. Thus,

$$|V(G')| = p + \left\lfloor \frac{p - 3b}{2} \right\rfloor = 3 \left\lfloor \frac{p - b}{2} \right\rfloor,$$

and

$$\delta(G') \geq \min(p, \delta(G) + \left\lfloor \frac{p - 3b}{2} \right\rfloor)$$

$$\geq \min(p, \left\lfloor \frac{p + b}{2} \right\rfloor + \left\lfloor \frac{p - 3b}{2} \right\rfloor)$$

$$= \min(p, 2 \left\lfloor \frac{p - b}{2} \right\rfloor)$$

$$= 2 \left\lfloor \frac{p - b}{2} \right\rfloor$$

$$= \frac{2}{3}(3 \left\lfloor \frac{p - b}{2} \right\rfloor)$$

$$\geq \frac{2}{3}(|V(G')| - 1).$$

Thus, by Theorem 9.2, either $H'$ is a subgraph of $G'$, or $G'$ is a graph of type 1 or type 2. Suppose $G'$ is a graph of type 2. If $G' \not\subseteq G$, then $G'$ has a vertex of degree $p$ and $|V(G')| < \frac{3}{2}p$. Hence, $G'$ is not a graph.
of type 2 unless $G' = G$. Then by Lemma 9.4,

$$|V(G')| = 4 \mod 6.$$  

In this case $\left\lfloor \frac{p - 3b}{2} \right\rfloor = 0$ vertices were added to $G$ to get $G'$, whence $p - 3b = 1$, and $H$ has $b = \frac{1}{3}(p - 1)$ triangles, and we have the second case of Theorem 9.1.

Suppose $G'$ is a graph of type 1. Then $H'$ has

$$b' = b + \left\lfloor \frac{p - 3b}{2} \right\rfloor = \left\lfloor \frac{p - b}{2} \right\rfloor$$

triangles. Moreover, $|V(G')| = 3b' + 1$, and there is a set $S' \subseteq V(G')$, with $|S'| = b' - 1$, whose removal leaves a complete bipartite graph $G' - S' = K_{b' + 1,b' + 1}$. We have

$$\delta(G') \geq 2 \left\lfloor \frac{p - b}{2} \right\rfloor = 2b'.$$

We claim that $V(G') = V(G) \cup S'$. To prove this, suppose that $V(G) \cup S'$ does not contain a vertex $v \in V(G') - V(G)$. However, $V(G') - V(G)$ has only $\left\lfloor \frac{p - 3b}{2} \right\rfloor$ vertices, and so some vertex $w$ of $G$ lies on the same side of the bipartition as $v$. But $v$ is adjacent to all vertices of $G$, and in particular to $w$, and we have a contradiction, which proves the claim.

Let $S = V(G) \cap S'$. Then, by the claim,

$$|S| = |S'| - (|V(G')| - |V(G)|)$$

$$= (b + \left\lceil \frac{p - 3b}{2} \right\rceil - 1) - \left\lfloor \frac{p - 3b}{2} \right\rfloor$$

$$= b - 1,$$

and $G - S$ is bipartite. This is a conclusion of 9.1.
The remaining possibility is that $H'$ is a subgraph of $G'$. There is an embedding of $H'$ into $G'$ which extends an embedding of $H$ into $G$. This proves Theorem 9.1.

**Lemma 9.5** Let $G$ be a graph, and $X_1 \cup X_2$ be a partition of $V(G)$ of the type described in Theorem 4.5, for which

\[(9.3) \quad \delta(G_1) + \delta(G_2) = \delta(G).\]

Suppose that sets $Y_3 \subseteq X_1$, $V_3 \subseteq X_2$ exist such that

\[(9.4) \quad G_2 - V_3 \text{ is a complete bipartite graph with nontrivial bipartition } V_1 \cup V_2;\]

\[(9.5) \quad G_1 - Y_3 \text{ is a complete bipartite graph with nontrivial bipartition } Y_1 \cup Y_2;\]

\[(9.6) \quad \text{If } v \in V_1 \cup V_2 \text{ then } \deg_{G_2}(v) = \delta(G_2);\]

\[(9.7) \quad \text{If } y \in Y_1 \cup Y_2 \text{ then } \deg_{G_1}(y) = \delta(G_1).\]

Then any vertex of $Y_1 \cup Y_2$ is adjacent to every vertex in $V_j$, for some $j \in \{1, 2\}$. Suppose further that

\[(9.8) \quad \text{No vertex of } Y_1 \cup Y_2 \text{ is adjacent to vertices in both } V_1 \text{ and } V_2.\]

Then $G - (Y_3 \cup V_3)$ is a complete bipartite graph.

**Proof:** By (9.3), (9.6), and (9.7), the latter part of Theorem 4.5 may be applied to the vertices of $V_1 \cup V_2 \cup Y_1 \cup Y_2$. 
Suppose that the first conclusion of the lemma is false for some \( y \in Y_1 \cup Y_2 \). Thus, \( y \) is not adjacent in \( G \) to a vertex \( v_1 \) of \( V_1 \) and a vertex \( v_2 \) of \( V_2 \). By Theorem 4.5, \( v_1 \) and \( v_2 \) are interchangeable with \( y \), and are thus not adjacent in \( G \). But, by (9.4), \( v_1 \) is adjacent to \( v_2 \). This contradiction proves the first part of the lemma.

By (9.8), any vertex \( y \in Y_1 \cup Y_2 \) is adjacent in \( G^c \) to every vertex of \( V_j \) for \( j \in \{1,2\} \). By the first part of the lemma, which was just proved, \( y \) is adjacent to every vertex of \( V_{j-1} \).

Thus, the vertices of \( Y_1 \cup Y_2 \) fall into two classes: those, the set of which we denote \( Y_4 \), which are adjacent to vertices of \( V_1 \) but not \( V_2 \); and those the set of which we denote \( Y_5 \), which are adjacent to vertices of \( V_2 \) but not \( V_1 \).

We claim that \( \{Y_4, Y_5\} = \{Y_1, Y_2\} \). To see this, suppose that \( Y_4 \cup Y_1 \) and \( Y_4 \cup Y_2 \) are both nonempty. Then any vertex \( v_2 \in V_2 \) is not adjacent to a vertex \( y_1 \in Y_4 \cup Y_1 \), nor to a vertex \( y_2 \in Y_4 \cup Y_2 \). By Theorem 4.5, \( y_1 \) and \( y_2 \) are interchangeable with \( v_2 \) and are thus not adjacent.

However, (9.5) implies that \( y_1 \) and \( y_2 \) are adjacent.
This contradiction shows that either \( Y_4 \setminus Y_1 \) or \( Y_4 \setminus Y_2 \) is empty. Similarly, either \( Y_5 \setminus Y_1 \) or \( Y_5 \setminus Y_2 \) is empty. Since \( V_1 \cup V_2 \) and \( Y_1 \cup Y_2 \) are nontrivial, and
\[
Y_4 \setminus Y_5 = Y_1 \setminus Y_2,
\]
the claim must follow.

In either case of this claim, there is a \( j \in \{1, 2\} \) such that \((V_1 \setminus Y_j) \cup (V_2 \setminus Y_{3-j})\) is a bipartition of \( G \setminus (Y_2 \setminus V_3) \), and this bipartite graph is complete. This proves Lemma 9.5.

We define for \( X_j' \subseteq V(G) \)
\[
G_j' = G[X_j'] \quad j = 1, 2,
\]
and
\[
p_j' = |X_j'| \quad j = 1, 2.
\]

A vertex \( x \) of \( G \), \( G_j \) or \( G_j' \) is critical in \( G \), \( G_j \), \( G_j' \), if
\[
\deg_G(x) - 1 < \frac{1}{3}(p - 1),
\]
\[
\deg_{G_j}(x) - 1 < \frac{1}{3}(p_j - 1),
\]
or
\[
\deg_{G_j'}(x) - 1 < \frac{1}{3}(p_j' - 1),
\]
respectively.
Lemma 9.6 Suppose Theorem 9.2 is valid for all graphs with less than \( p \) vertices. Suppose
\[
p \equiv 1 \pmod{3}
\]
and that \( X_1 \cup X_2 \) is a partition of \( V(G) \) which satisfies the conditions of Theorem 4.5 with \( c = \frac{2}{3} \). For \( \{z,z'\} \subseteq V(G) \), write
\[
X_j' = X_j - \{z,z'\} \quad \text{for} \; j = 1,2,
\]
and assume that
\[
p_j' \equiv 1 \pmod{3} \quad \text{for} \; j = 1,2.
\]
(9.9) \( \delta(G') \geq \frac{2}{3}(p_j' - 1) \) for \( j = 1,2 \), and that \( p_j \equiv 0 \pmod{3} \) for \( j \in \{1,2\} \) implies that \( z \in X_j \) and that there exist critical vertices \( x_3, x_4 \in X_{3-j} \) such that \( G[z,x_3,x_4] \) is a triangle. Then, if \( \frac{1}{3}(p_j' - 1) \) pairwise disjoint triangles cannot be embedded in \( G_j' \), for \( j = 1 \) and \( j = 2 \), both \( G_1' \) and \( G_2' \) are of type 1.

Proof: Since Theorem 9.2 holds for graphs on fewer than \( p \) vertices, since (9.9) holds, and since \( \frac{1}{3}(p_j' - 1) \) triangles cannot be embedded in \( G_j' \), \( j = 1,2 \), it follows that \( G_j' \) is of type 1 or type 2, and \( G_2' \) is of type 1 or type 2. Thus,
\[
\delta(G_j') = \frac{2}{3}(p_j' - 1) \quad j = 1,2,
\]
whence
\[
(9.10) \quad \delta(G_1') + \delta(G_2') = \frac{2}{3}(p_1' + p_2' - 2)
\]
\[
= \frac{2}{3}(p - 1) - 2.
\]
Moreover, by Theorem 4.5,
\[ \delta(G_1) + \delta(G_2) \geq \frac{2}{3}(p-1) - \frac{2}{3}. \]
The left side is an integer and \( p \equiv 1 \pmod{3} \), whence
\[ \delta(G_1) + \delta(G_2) \geq \frac{2}{3}(p-1). \]
So that (9.10) also holds, it follows that if \( z \) or \( z' \), respectively, is in \( X_j \), for \( j \in \{1,2\} \), then \( z \) or \( z' \) is adjacent in \( G \) to every critical vertex of \( G'_j \). In fact
\[ (9.11) \quad \delta(G_1) + \delta(G_2) = \frac{2}{3}(p-1), \]
and vertices critical in \( G'_j \) are critical in \( G_j \), for \( j = 1,2 \). Also, since critical vertices of \( G_1 \) and critical vertices of \( G_2 \) are interchangeable if they are adjacent in \( G^c \), critical vertices of \( G'_1 \) and critical vertices of \( G'_2 \) are also interchangeable if they are adjacent in \( G^c \). By (4.19) of Theorem 4.5, such vertices are also critical in \( G \).

Let \( y_1, y_2 \) be any pair of adjacent critical vertices of \( G'_1 \). If \( G'_1 \) is of type 2, then every vertex of \( G'_1 \) is critical in \( G'_1 \), whence, any adjacent pair suffices. If \( G'_1 \) is of type 1, then \( p'_1 \geq 4 \), and there is a set \( Y_3 \subseteq X'_1 \) with
\[ |Y_3| = \frac{1}{3}(p'_1 - 1) - 1, \]
such that \( G'_1 - Y_3 \) is a complete bipartite graph with bipartition \( Y_1 \cup Y_2 \), where
\[|Y_1| = |Y_2| = \frac{1}{3}(p'_j - 1) + 1.\]

Since \(Y_1 \cap Y_2\) is the set of critical vertices in \(G'_1\), if \(y_1 \in Y_1\) and \(y_2 \in Y_2\), then \(y_1\) and \(y_2\) are adjacent critical vertices of \(G'_1\).

Suppose by way of contradiction that \(G'_2\) is of type 2 and not of type 1. Then every vertex \(v\) of \(G'_2\) is critical in \(G'_2\), and hence interchangeable with \(y_1\) \((1 = 1, 2)\) if \(y_1\) is adjacent in \(G^c\) to \(v\).

Since \(y_1\) is critical in \(G'_1\) and in \(G\),

\[
|E(y_1, X'_2)| = \deg_G(y_1) - \deg_{G'_1}(y_1) - |E(y_1, \{z, z'\})| = \frac{2}{3}(p - 1) - \frac{2}{3}(p'_1 - 1) - |E(y_1, \{z, z'\})| = \frac{2}{3}(p'_2 + 2) - |E(y_1, \{z, z'\})|.
\]

Hence, the number of vertices of \(G'_2\) adjacent in \(G^c\) to \(y_1\) is at least

\[
p'_2 - |E(y_1, X'_2)| \geq \frac{1}{3}(p'_2 - 1) + |E(y_1, \{z, z'\})| - 1,
\]

and these vertices are interchangeable with \(y_1\) and thus form a stable set.

We have two cases: when \(\{z, z'\} \cap X_1\) is not empty, and when \(\{z, z'\} \subseteq X_2\). In the first case, without loss of generality, suppose \(z \in X_1\). In \(G_1\), \(z\) is adjacent to every critical vertex of \(G'_1\), including \(y_1, y_2 \in X'_1\). Hence, \(|E(y_1, \{z, z'\})| \geq 1\). In the second case, by the hypotheses of the lemma, \(p_2 \equiv 0 \pmod{3}\) and \(z\) lies in a triangle \(G[z, x_3, x_4]\), where \(x_3\) and \(x_4\) are adjacent.
critical vertices of \(G_1\). Pick \(y_1, y_2\) so that
\[\{y_1, y_2\} = \{x_3, x_4\}\], which is possible because \(G_1 = G_1'\)
here. Then \(|E(y_1, iz, z')| \geq 1\). Therefore, in either
case there are at least \(\frac{1}{3}(p_2' - 1)\) critical vertices of
\(G_2'\) interchangeable with \(y_1\) \((i = 1, 2)\).

Since \(G_2'\) is of type 2, \(p_2' = 4 \pmod{6}\), and since
\(G_2'\) is not of both type 1 and type 2, it follows that
\(p_2' \geq 10\). Hence, at least \(\frac{1}{3}(p_2' - 1) \geq 3\) critical vertices
of \(G_2'\) are interchangeable with \(y_1\) \((i = 1, 2)\). By Theorem
4.5, this set of \(\frac{1}{3}(p_2' - 1)\) vertices is a stable set.
But since \(G_2'\) is of type 2, there is only one maximal
stable set \(S_2\) of more than 2 vertices, and \(S_2\) has
\(\frac{1}{3}(p_2' - 1) + 1\) vertices. Therefore, \(y_1\) is interchangeable
with all but at most one vertex of \(S_2\). Since \(|S_2| \geq 3\),
there is a critical vertex \(v \in S_2\), critical in \(G_2'\),
interchangeable with both vertices \(y_1, y_2\) critical in \(G_1\).
By Theorem 4.5, \(y_1\) and \(y_2\) are not adjacent, contrary to
the choice of \(y_1\) and \(y_2\). Hence, \(G_2'\) is of type 1, and
the lemma is proved.
We leave to the reader the proofs of the next two lemmas.

**Lemma 9.7** Let $G_0$ be a graph of type 1 on $3b_0 + 1$ vertices. Let $S_0$ be the set of $b_0 - 1$ vertices whose removal leaves $G - S_0 = K_{b_0+1,b_0+1}$. Any embedding of $b_0 - 1$ pairwise disjoint triangles into $G_0$ uses all but four vertices $v_1, v_2, v_3, v_4 \in V(G_0) - S_0$, and these four vertices induce a quadrilateral in $G_0$. Furthermore, $v_1, v_2, v_3, v_4$ may be chosen to be any four vertices of $G_0 - S_0$ that induce a quadrilateral in $G$.

**Lemma 9.8** Let $G_0$ be a graph of type 2 on $3b_0 + 1$ vertices. Let $S_0$ be the stable set of $b_0 + 1$ vertices such that $G_0 - S_0$ consists of two components, each $K_{b_0}$. Any embedding of $b_0 - 1$ pairwise disjoint triangles into $G_0$ uses all but four vertices, two in $S_0$, and one in each $K_{b_0}$, and these four vertices induce a quadrilateral in $G_0$. Furthermore, for any four vertices of $V(G_0)$ with two in $S_0$ and one in each $K_{b_0}$, there is an embedding of $b_0 - 1$ pairwise disjoint triangles into the remaining $3b_0 - 3$ vertices of $G_0$. 

To save work, we assume without proof the following result of Corradi and Hajnal [7]:

Theorem 9.9 Let G and H be graphs on p vertices such that every component of H is a triangle, except possibly for one component that is either $K_1$ or $K_2$. If

$$\delta(G) \geq \frac{2p-1}{3},$$

then H is a subgraph of G.

Proof of Theorem 9.2: By Theorem 9.9, it suffices to consider graphs G for which

$$\delta(G) = \frac{2}{3}(p-1),$$

Equality implies that

$$p \equiv 1 \pmod{3}.$$ 

Thus, we can assume that H is a graph with b triangles and one isolated vertex, and that

$$p = 3b + 1,$$

$$\delta(G) = \frac{2}{3}(p-1) = 2b.$$ 

By Theorem 4.5 there are disjoint nonempty sets $X_1, X_2$ such that $V(G) = X_1 \cup X_2$ and the induced subgraphs $G_i$, for $G_i = G[X_i]$, $i = 1, 2$, satisfy

$$(9.12) \quad \delta(G_i) \geq \frac{2}{3}(p_i - 1),$$

where $p_1 = |X_1|$. 

Assume inductively that Theorem 9.2 is true for graphs smaller than G, and suppose that H is not a
subgraph of $G$. Theorem 9.2 is true for $p \leq 4$, and so we have a basis for induction. We have two cases: either one of the sets $X_1$ has cardinality a multiple of 3, or neither do. In one subcase (Subcase II), we show that if $H$ is not a subgraph of $G$, then $G$ is of type 2. In other subcases, we verify the hypotheses of Lemma 9.5, and hence there is a subset $S = Y_3 \cup V_3$ of $V(G)$, with $|S| = b - 1$, such that $G - S$ is a bipartite graph. Thus, by Lemma 9.3, $G$ is of type 1. We consider each case below.

**Case I:** Suppose that

$$p_1 \equiv 0 \pmod{3}$$

and

$$p_2 \equiv 1 \pmod{3}.$$

Since $\delta(G_1)$ is an integer and $p_1 \equiv 0 \pmod{3}$, (9.12) gives

$$\delta(G_1) \geq \frac{2}{3}(p_1 - 1) + \frac{2}{3} = \frac{2}{3}p_1,$$

and Theorem 9.9 implies that $p_1/3$ triangles can be embedded in $G_1$. Write

$$b_1 = \frac{1}{3}p_1$$

and

$$b_2 = \frac{1}{3}(p_2 - 1),$$

and note that

$$b_1 + b_2 = b.$$
and that the $b_1 \geq 1$ triangles embedded in $G_1$ use each vertex of $G_1$. Since $b$ triangles are assumed to not embed in $G$, it follows that $b_2$ triangles do not embed in $G_2$. By the induction hypothesis, either $G_2$ is of type 1, and there is a set $V_3 \subseteq X_2$ with

\begin{equation}
|V_3| = b_2 - 1
\end{equation}

such that

$$G_2 - V_3 = K_{b_2 + 1, b_2 + 1},$$

or $G_2$ is of type 2 and there is a stable set

\begin{equation}
S_2 \subseteq X_2
\end{equation}

such that $G_2 - S_2$ has two components, each a clique on $b_2$ vertices.

If $G_2$ is of type 2, each vertex $v \in X_2$ has degree

$$\deg_{G_2}(v) = 2b_2 = \frac{2}{3}(p_2 - 1),$$

and so (9.6) holds. If this alternative applies, write

\begin{equation}
V_2 = S_2, \quad V_1 = G_2 - S_2.
\end{equation}

If $G_2$ is of type 1, let

\begin{equation}
V_1 \circ V_2 \text{ denote the bipartition of } G_2 - V_3.
\end{equation}

Then

$$|V_1| = |V_2| = b_2 + 1,$$

and (9.12) and (9.13) give

$$\delta(G_2) \geq 2b_2,$$

which allows us to apply Lemma 9.3. Also, by Lemma 9.3,
each vertex of $V_j$ ($j=1,2$) is adjacent to every vertex of $V_{3-j}$ and to every vertex of $V_3$, and if $v \in V_1 \cup V_2$,
\[
\deg_{G_2}(v) = 2b_2 = \frac{2}{3}(p_2 - 1)
\]
whence (9.6) holds.

It follows in either alternative (either $G_2$ of type 1 or type 2) that there must be at least
\[
\deg_G(v) - \deg_{G_2}(v) \geq \frac{2}{3}(p - 1) - \frac{2}{3}(p_2 - 1)
\]
\[
= \frac{2}{3}p_1
\]
vertices in $X_1$ adjacent to a given vertex $v \in V_1 \cup V_2$.

Denote by $N(v_1,v_2)$ the vertices of $X_1$ that are adjacent to both $v_1 \in V_1$ and $v_2 \in V_2$. We have

(9.18) $|N(v_1,v_2)| \geq 2\left(\frac{2}{3}p_1\right) - p_1 = \frac{1}{3}p_1 = b_1$.

Since $G_2$ is of type 1 or type 2, $b_2$ disjoint triangles do not embed in $G_2$. By Lemmas 9.7 and 9.8, there is an embedding of $b_2 - 1$ pairwise disjoint triangles into $G_2$ such that the four remaining vertices induce a quadrilateral in $G_2$, with two of its vertices in $V_1$ and the other two in $V_2$. Let $\{v_1,v_2\}$ and $\{v'_1,v'_2\}$ be disjoint edges of this quadrilateral, where $v_1,v'_1 \in V_1$, and $v_2,v'_2 \in V_2$. 
In the two subcases below, we establish that $G_2$ is of type 1, and that the hypotheses of Lemma 9.5 apply to $G_1$ and $G_2$. We have already established (9.4) and (9.6), and it remains to establish (9.3), (9.5), and (9.7). After the subcases, we prove (9.8).

**Subcase IA:** Suppose $v_1, v'_1 \in V_1$ are distinct and $v_2, v'_2 \in V_2$ are distinct. Suppose that $N(v_1, v_2)$, $N(v'_1, v'_2)$ possess a transversal $\{y, y'\}$ in $X_1$; i.e., distinct $y, y' \in X_1$ such that

$$y \in N(v_1, v_2), \quad y' \in N(v'_1, v'_2)$$

Since

$$\delta(G_1) \geq \frac{2}{3} p_1,$$

we have

$$\delta(G_1 - \{y, y'\}) \geq \frac{2}{3} p_1 - 2 = \frac{2}{3} (p_1 - \{|y, y'\} - 1).$$

Since $b$ pairwise disjoint triangles do not embed in $G$, and since $(b_2 - 1) + 2$ triangles can be embedded in $G[X_2 \setminus \{y, y'\}]$, we cannot embed

$$b - (b_2 + 1) = b_1 - 1$$

triangles in $G_1 - \{y, y'\}$. By the induction hypotheses, $G_1 - \{y, y'\}$ is a graph of type 1 or of type 2, and by Lemma 9.6 with $\{y, y'\} = \{z, z'\}$, and with $\{v_1, v_2\} = \{x_3, x_4\}$, both $G_1 - \{y, y'\}$ and $G_2$ are of type 1. Therefore, there is a set $Y'_3$ of $b_1 - 2$ vertices such that
\( G_1 - Y_3 \) is bipartite, where

\[ (9.19) \quad Y_3 = Y_3^1 \cup \{y, y'\}. \]

Let \( Y_1 \sim Y_2 \) be the bipartition of \( G_1 - Y_3 \). By definition,
\[ |Y_1| = |Y_2| = b_1 = \frac{1}{3} p_1, \]
and by Lemma 9.3, each vertex \( y_j \) of \( Y_j \) \( (j=1,2) \) is adjacent to every vertex of \( Y_3 - j \cup Y_j' \) and has degree
\[ \frac{2}{3} p_1 - 2 \text{ in } G_1 - \{y, y'\}. \]
Thus, (9.5) holds. Since
\[ \deg_{G_1}(y_j) \geq \delta(G_1) = \frac{2}{3} p_1, \]
each vertex of \( Y_j \) is also adjacent to \( y \) and \( y' \), and hence has degree \( \delta(G_1) \) in \( G_1 \), whence we have (9.7). Therefore,
\[
\delta(G_1) + \delta(G_2) = \frac{2}{3} p_1 + \frac{2}{3} (p_2 - 1)
= \frac{2}{3} (p - 1)
= \delta(G),
\]
which is (9.3). We have thus proved (9.3) and (9.4) through (9.7) of Lemma 9.5. This completes Subcase IA.

Subcase IB: Suppose that there is no pair of disjoint edges \( \{v_1, v_2\}, \{v_1', v_2'\} \) in \( G_2[V_1 \sim V_2] \) such that \( N(v_1, v_2), N(v_1', v_2') \) possess a transversal.

Since \( p_1 > 0 \), (9.18) implies that \( b_1 \geq 1 \) and that \( N(v_1, v_2) \) and \( N(v_1', v_2') \) are nonempty. Since \( N(v_1, v_2) \) and \( N(v_1', v_2') \) possess no transversal, we have \( y \in X_1 \) such that \( N(v_1, v_2) = y = N(v_1', v_2') \).
Hence, $x_1$ is not adjacent to itself in $G_1$, nor to
\[ \frac{1}{3}(p-1) = \frac{1}{3}(p_2-1) + 1 \]
vertices in $X_2$, which, by Theorem 4.5, must be a stable set. Since $G_2$ is of type 2, there is only one stable set, namely $S_2$, by (9.15), of
\[ b_2 + 1 = \frac{1}{3}(p_2-1) + 1 \]
vertices, unless $b_2 + 1 = 2$. If $b_2 = 1$, then $p_2 = 4$, and $G_2$ is a quadrilateral, which is also of type 1. If $b_2 \geq 2$, each vertex of $X_1$ is interchangeable with any vertex of $S_2$, and since they are interchangeable, Theorem 4.5 implies that $X_1$ is stable. This contradicts the fact that $G_1$ is a triangle. Hence, $G_2$ is of type 1.

Finally, we must show that (9.8) of Lemma 9.5 applies in either subcase. Let $y$, $N(v_1, v_2)$, and $N(v'_1, v'_2)$ be as in the subcases above. Suppose (9.8) is false.

There exists a vertex $y'' \in Y_1 \cup Y_2$ that forms a triangle with vertices of $V_1 \cup V_2$. Then the first part of Lemma 9.4 implies that $y''$ is adjacent to all vertices of $V_j$ and to some of the vertices of $V_{3-j}$, for $j = 1$ or 2. Choose
\[ v = v_{3-j} \text{ or } v'_{3-j} \]
so that \( y'' \) is adjacent to a vertex \( v_{3-j} \) of \( V_{3-j} - v \).

Without loss of generality, suppose
\[ v = v_{3-j}. \]

Then, \( v_{3-j}, v_j \) and \( y \) form a triangle, and \( y'' \) forms a triangle with \( v_j' \) and \( v_{3-j}' \). Thus, there are two disjoint triangles, which together with the \( b_2 - 1 \) triangles that can be embedded in \( G_2 - \{v_j, v_j', v_{3-j}, v_{3-j}' \} \), and the \( b_1 - 1 \) triangles that can be embedded in \( G_1 - \{y'', y'\} \) constitute an embedding of
\[ 2 + b_2 - 1 + b_1 - 1 = b \]
triangles in \( G \). We have contradicted the nonembeddability assumption, and hence (9.8) is true. Thus, all of the hypotheses of Lemma 9.5 hold. We conclude from Lemma 9.5 that \( G - (Y_3 \cup V_3) \) is a bipartite graph.

By (9.14), (9.19), (9.20), we have
\[ |Y_3 \cup V_3| = b - 1, \]
and so by Lemma 9.3, \( G \) is of type 1. This completes Case I.
Case II: Suppose that
\[ p_1 \equiv p_2 \equiv 2 \pmod{3}. \]
Since \( \delta(G_1) \) is an integer, (9.12) implies
\[ \delta(G_1) \geq \frac{2}{3}p_1 - \frac{1}{3} \tag{9.22} \]
for \( i = 1, 2 \). Without loss of generality, assume
\[ b_1 \leq b_2. \]
Write
\[ b_1 = \frac{1}{3}p_1 - \frac{2}{3}, \quad b_2 = \frac{1}{3}p_2 - \frac{2}{3}, \]
and note that \( b_1 \) and \( b_2 \) are integers such that
\[ b_1 + b_2 + 1 = b. \]
If we form a graph \( G_1 + z \), adding to \( G_1 \) (\( i = 1, 2 \)) a new vertex \( z \) adjacent to every vertex of \( G_1 \), then by (9.22),
\[ \delta(G_1 + z) = \frac{2}{3}|x_1 + z|, \]
and by Theorem 9.9, \( b_1 + 1 \) pairwise disjoint triangles can be embedded in \( G_1 + z \). Therefore, \( b_1 \) pairwise disjoint triangles and an edge disjoint from the \( b_1 \) triangles, which we shall call the free edge, can be embedded in \( G_1 \). We shall attempt to use the vertices of the two free edges to form an extra triangle, disjoint from the \( b_1 \) triangles in \( G_1 \) and the \( b_2 \) triangles in \( G_2 \), thus constituting \( b_1 + b_2 + 1 = b \) pairwise disjoint triangles in \( G \). By assuming that \( b \) pairwise disjoint
triangles do not embed in \( G \), we shall determine the structure of \( G \) in the attempt to find such an embedding.

We show in the two subcases below that either \( G \) is of type 2, or there is a vertex \( x_3 \in X_2 \) such that the free edge in \( G_1 \) together with \( x_3 \) form a triangle in \( G \). It may be necessary to alter the embedding of \( b_1 \) triangles and the free edge into \( G_1 \) in order to accomplish this.

Let \( x_1, x_2 \) be the ends of the free edge in \( G_1 \). Without loss of generality, choose the free edge from among all possible free edges so that

\[
\deg_{G_1}(x_1) + \deg_{G_1}(x_2)
\]

is minimized. If \( x_1 \) and \( x_2 \) are adjacent in \( G \) to a vertex \( x_3 \in X_2 \), then \( x_1, x_2, x_3 \) is the desired triangle. Otherwise, \( x_1 \) and \( x_2 \) are adjacent to no common vertex in \( X_2 \). Then

\[
(9.23) \quad \deg_{G_1}(x_1) + \deg_{G_1}(x_2) \geq 2\delta(G) - p_2 \\
\geq \frac{4}{3}(p - 1) - p_2 \\
= p_1 + \frac{1}{3}p - \frac{4}{3}.
\]

Also, without loss of generality, assume that

\[
\deg_{G_1}(x_1) \geq \deg_{G_1}(x_2).
\]

These inequalities imply

\[
(9.24) \quad 2\deg_{G_1}(x_1) \geq \deg_{G_1}(x_1) + \deg_{G_1}(x_2) \\
\geq p_1 + \frac{1}{3}p - \frac{4}{3}.
\]
We define 
\[ \pi: V(H_1) \to V(G_1) \]
to be an embedding of \( b_1 \) triangles \( K_3 \) and one edge-component \( K_2 \), constituting \( H_1 \), into \( G_1 \) such that the edge-component \( K_2 \) is mapped to the free edge \( x_1, x_2 \) that minimizes \( \deg_{G_1}(x_1) + \deg_{G_1}(x_2) \). We shall alter \( \pi \) if necessary, and then either we shall extend \( \pi \) to an embedding of \( H \) into \( G \), where \( H \) consists of \( b \) triangular components and one isolated vertex, or we shall show (Subcase IIA) that \( G \) is of type 2 or (following the subcases) that \( G \) is of type 1.

Define 
\[ M(x) = \{ x' \in X_1 : \pi^{-1}(x) \text{ and } \pi^{-1}(x') \text{ are adjacent in } H_1 \} \].

For \( i = 1, 2 \), and \( x \in V(G) \), define 
\[ N_i(x) = \{ x' \in X_i : x \text{ and } x' \text{ are adjacent in } G_i \} \].

We say that \( x \in X_1 \) is a **successor** of \( x_1 \in X_1 \) if each vertex of \( M(x) \) is adjacent in \( G_1 \) to \( x \). Denote the set of successors of \( x_1 \) by \( S(x_1) \). We say that \( x_1 \in X_1 \) is a **predecessor** of \( x \in X_1 \) if \( x \) is a successor of \( x_1 \). Denote the set of predecessors of \( x \) by \( P(x) \).

Recall from section 1 that if \( x_1, x_4 \in X_1 \) are equal, then \( (x_1, x_4) \) is the identity permutation on \( X_1 \), but if \( x_1, x_4 \) are distinct, then \( (x_1, x_4)^r = (x_1, x_4) \).
Subcase IIA: Suppose that 

\[ \deg_{G_1}(x_2) \leq \frac{1}{3}(p-1). \]

First, we eliminate the possibility of strict inequality. If the inequality above is strict, then

\[ |E(x_2, x_2)| = \deg_G(x_2) - \deg_{G_1}(x_2) \]
\[ > \frac{2}{3}(p-1) - \frac{1}{3}(p-1) \]
\[ = \frac{1}{3}(p-1). \]

Since \( x_1 \) is not adjacent to at most \( \frac{1}{3}(p-1) \) vertices of \( G \) other than itself, \( x_1 \) is adjacent to one of the more than \( \frac{1}{3}(p-1) \) vertices \( x_3 \) of \( X_2 \) incident with an edge of \( E(x_2, x_2) \). Hence, \( G[x_1, x_2, x_3] \) is a triangle on the free edge in \( G_1 \) and a vertex of \( G_2 \).

Henceforth in this subcase, we shall suppose

\[ \deg_{G_1}(x_2) = \frac{1}{3}(p-1). \]

By (9.23),

\[ \deg_{G_1}(x_1) + \frac{1}{3}(p-1) \geq p_1 + \frac{1}{3}p - \frac{4}{3}. \]

Hence,

\[ \deg_{G_1}(x_1) \geq p_1 - 1, \]

and so \( x_1 \) must be adjacent to each vertex of \( G_1 \).

Therefore, \( P(x_1) = G_1 - x_2 \). Since \( S(x_1) = N_1(x_2) \), we conclude that for any \( x_4 \in N_1(x_2) \), \( (x_1 x_4)^\pi \) is an embedding of the \( b_1 \) triangles and free edge into \( G_1 \). Note that
the embedding \((x_1, x_4)^n\) makes \(\{x_4, x_2\}\) the free edge.

By the minimality of \(\deg_{G_1}(x_1) + \deg_{G_1}(x_2)\),

\[
\deg_{G_1}(x_4) + \deg_{G_1}(x_2) \geq \deg_{G_1}(x_1) + \deg_{G_1}(x_2),
\]

whence,

\[
\deg_{G_1}(x_4) = p_1 - 1.
\]

Since \(x_4\) may be any of the \(\frac{1}{3}(p - 1)\) vertices of \(N_1(x_2)\),
we know that the vertices of \(X_1 - N_1(x_2)\) must be adjacent to each vertex of \(N_1(x_2)\), a set of \(\frac{1}{3}(p - 1)\) vertices adjacent to all of \(G_1\). Hence,

\[
\delta(G_1) \geq \frac{1}{3}(p - 1) = \deg_{G_1}(x_2).
\]

Define the sets

\[
T_1 = N_1(x_2),
\]

\[
T_2 = N_2(x_2) ,
\]

\[
S_1 = X_1 - T_1,
\]

\[
S_2 = X_2 - T_2.
\]

We have already shown that \(G[T_1]\) is a complete graph,
and each vertex of \(S_1\) is adjacent to every vertex of \(T_1\).

If there is an \(x_4 \in T_1 = S(x_1)\) and a vertex \(x_3 \in X_2\) such
that \(G[x_2, x_3, x_4]\) is a triangle in \(G\), then we have accomplished
the goal of this subcase, since \((x_1, x_4)^n\) is an embedding
of \(b_1\) triangles and a disjoint edge mapped to \(\{x_2, x_4\}\),
which is the edge forming the triangle with \(x_3\).

Otherwise, no \(x_4 \in T_1\) forms a triangle with \(x_2\) and any
vertex in \(X_2\). Hence, no \(x_4 \in T_1\) is adjacent to vertices
of $T_2$. Now,

$$|T_2| = \deg_G(x_2) - \deg_{G_1}(x_2) \geq \frac{1}{3}(p - 1),$$

and hence, any $x_4 \in T_1$, having degree at least $\frac{2}{3}(p - 1)$ in $G$, must be adjacent to every vertex of $S_1 \cup T_1 \cup S_2 - x_4$. A similar argument shows that any vertex of $T_2$, not being adjacent to any vertex of $T_1$, a set of $\frac{1}{3}(p - 1)$ vertices, is adjacent to any vertex of $S_1 \cup T_2 \cup S_2$ except itself. Note that this implies that $G[T_2]$ is, like $G[T_1]$, a complete graph on $\frac{1}{3}(p - 1)$ vertices.

Also, note that any vertex of $S_1 \cup S_2$ is adjacent to every vertex of $T_1 \cup T_2$ in $G$.

Hence, $S_1 \cup S_2$ is a set of

$$|V(G) - (T_1 \cup T_2)| = p - \frac{2}{3}(p - 1)$$

$$= \frac{1}{3}(p - 1) + 1$$

$$= b + 1$$

vertices whose removal from $G$ leaves two components $G[T_i]$, $i = 1, 2$, each a complete graph on $\frac{1}{3}(p - 1) = b$ vertices.

By Lemma 9.4, either $b$ pairwise disjoint triangles embed in $G$, or $G$ is of type 2. The first possibility is contrary to hypothesis. The other possibility is a desired conclusion of Theorem 9.1. Hence, we can assume that there is a free edge in $G_1$, which together with some $x_3 \in X_2$, forms a triangle in $G$. 
Subcase IIB: Suppose that
\[(9.25) \quad \deg_{G_1}(x_2) > \frac{1}{3}(p-1).\]
Let $x_3$ be a vertex of $X_2$ that is adjacent in $G$ to $x_2$.

Since $p_1 \leq p_2$,
\[
\deg_{G}(x_2) \geq \frac{2}{3}(p-1)
\geq \frac{2}{3}(2p_1 - 1)
= p_1 + \frac{1}{3}p_1 - \frac{2}{3}
> p_1 - 1,
\]
and so $x_3$ exists. The successors $S(x_1)$ of $x_1$ in $G_1$ are the vertices of $G_1$ adjacent to $x_2$. We see that $S_1(x_1) = N_1(x_2)$. We have
\[
|S(x_1) \cap N_1(x_3)| \geq \deg_{G_1}(x_2) + \deg_{G}(x_3)
- (p_2 - 1) - |S(x_1) \cup N_1(x_3)|
\geq \deg_{G_1}(x_2) + \frac{2}{3}(p-1) - (p_2 - 1) - p_1
= \deg_{G_1}(x_2) - \frac{1}{3}(p-1)
> 0,
\]
by (9.25). Hence, there is a vertex $x_4 \in X_1$ that forms a triangle with $x_2$ and $x_3$ and is a successor of $x_1$.

If $x_1 \in S(x_4)$, then the embedding \((x_1, x_4)\) maps the free edge in $G_1$ to \(\{x_2, x_4\}\), which forms with $x_3 \in X_2$ a triangle in $G$ as desired. Otherwise,
\[(9.26) \quad x_1 \notin S(x_4).\]
We shall find a vertex \( x_5 \in X_1 \) with \( x_5 \in S(x_4) \triangle P(x_1) \), whence \( (x_1 \, x_4 \, x_5) \) is the desired embedding of \( b_1 \) triangles and one edge into \( G_1 \).

In the image of the triangle embedded into \( G_1 \) having vertex \( x_4 \) are two other vertices, which we call \( x_6, x_7 \). The successors of \( x_4 \) are those vertices in \( G_1 \) adjacent to both \( x_6 \) and \( x_7 \). Hence, \( x_1, x_6, x_7 \notin S(x_4) \), and

\[
(9.27) \quad |S(x_4)| \geq \deg_{G_1}(x_6) + \deg_{G_1}(x_7) - p_1.
\]

The predecessors \( P(x_1) \) of \( x_1 \) in \( G_1 \) are those vertices \( v \in X_1 \) such that \( x_1 \) is adjacent to all vertices of \( M(v) \). Now, \( x_1 \) is adjacent in \( G_1^c \) to \( p_1 - \deg_{G_1}(x_1) - 1 \) vertices \( v' \in X_1 \). Any such \( v' \) lies in exactly two sets \( M(v), v \in X_1 \). Thus, \( x_1 \notin S(v) \) for at most

\[
2p_1 - 2\deg_{G_1}(x_1) - 2
\]

vertices \( v \) of \( X_1 - M(x_1) = G - x_2 \). Since the remaining vertices of \( G_1 - x_2 \) are in \( P(x_1) \), we have \( x_2 \notin P(x_1) \), and

\[
(9.28) \quad |P(x_1)| \geq |X_1 - x_2| - (2p_1 - 2\deg_{G_1}(x_1) - 2)
= 2\deg_{G_1}(x_1) - p_1 + 1
\geq \frac{1}{3}(p - 1),
\]

by (9.24).

Suppose first that \( x_4 \) is not adjacent to \( x_1 \). Then

\[
x_1, x_6, x_7 \notin P(x_1),
\]

and we combine (9.27), (9.28), (9.22), and \( 2p_1 \leq p \) to get

\[
x_1, x_2, x_6, x_7 \notin S(x_4) \cup P(x_1);
\]

\[
x_6, x_7 \notin S(x_4) \cup P(x_1).
\]
and

\[(9.29) \quad |S(x_4) - P(x_1)| \geq |S(x_4)| + |P(x_1)| - |x_1 - x_6, x_7|\]
\[\geq \deg_{G_1}(x_6) + \deg_{G_1}(x_7) - p_1\]
\[+ \frac{p}{3} - \frac{1}{3} - p_1 + 2\]
\[\geq 2\delta(G_1) - 2p_1 + \frac{p}{3} + \frac{5}{3}\]
\[\geq 2\left(\frac{2}{3}p_1 - \frac{1}{3}\right) - 2p_1 + \frac{p}{3} + \frac{5}{3}\]
\[= \frac{p}{3} - \frac{2}{3}p_1 + 1\]
\[\geq 1.\]

Suppose, otherwise, that $x_4$ is adjacent to $x_1$. Then $G[x_1, x_2, x_4]$ is a triangle, and $x_6, x_7$ is a free edge.

Thus, by choice of $x_1, x_2, x_4$ and (9.23),

\[(9.30) \quad \deg_{G_1}(x_6) + \deg_{G_1}(x_7) \geq \deg_{G_1}(x_1) + \deg_{G_1}(x_2)\]
\[\geq p_1 + \frac{p}{3} - \frac{4}{3}.\]

We combine (9.27), (9.28), (9.30) and

\[p_1 + p_2 = p\]

to obtain

\[(9.31) \quad |S(x_4) - P(x_1)| \geq |S(x_4)| + |P(x_1)| - p_1\]
\[\geq \deg_{G_1}(x_6) + \deg_{G_1}(x_7) - p_1\]
\[+ \frac{p}{3} - \frac{1}{3} - p_1\]
\[\geq p_1 + \frac{p}{3} - \frac{4}{3} - 2p_1 + \frac{p}{3} - \frac{1}{3}\]
\[= \frac{2p}{3} - \frac{2}{3}p_1 - \frac{1}{3}p_1 - \frac{5}{3}\]
\[\geq \frac{2}{3}p_2 - \frac{1}{3}p_1 - \frac{5}{3}\]
\[= \frac{1}{3}(p_2 - p_1) + (\frac{1}{3}p_2 - \frac{5}{3}).\]
Note that both of the terms in the last line of (9.31) are nonnegative if \( p_2 \geq 5 \), and if \( p_2 > 5 \), then the last line is positive. If \( p_2 \leq 5 \), then \( p_1 \leq p_2 \) and \( p_1 \equiv 2 \pmod{3} \) imply one of the following three cases:

\[
P_2 = p_1 = 5;
\]

\[
P_2 = 5, \quad p_1 = 2;
\]

or

\[
P_2 = p_1 = 2.
\]

If \( p_2 = p_1 = 5 \), then (9.23) gives

\[
\deg_{G_1}(x_1) + \deg_{G_1}(x_2) \geq 7,
\]

whence \( \deg_{G_1}(x_1) \geq \deg_{G_1}(x_2) \) implies that \( x_1 \) is adjacent to every vertex of \( G_1 \) except itself, whence \( x_4 \in P(x_1) \), in violation of (9.26). If \( p_2 = 5, \ p_1 = 2 \), then the last line of (9.31) is 1, which is as desired. If \( p_1 = p_2 = 2 \), then \( p = 4 \) and \( \delta(G) \geq \frac{2}{3}(p - 1) \) imply \( G \) is \( K_4, K_4 - e \) (e an edge), or a quadrilateral, all of which satisfy the theorem. Hence, under our hypotheses, the last line of (9.31) and the last line of (9.29) may be assumed to be positive.

Therefore, whether or not \( x_4 \) and \( x_1 \) are adjacent, there is a vertex \( x_5 \neq x_1 \) or \( x_2 \), such that

\[
x_5 \in S(x_4) \cap P(x_1),
\]

and so we have a closed alternating chain in \( G_1 \) represented
by the permutation 
\[ \alpha = (x_1 \ x_4 \ x_5). \]
Hence, \( \alpha \omega \) is an embedding of the \( b_1 \) triangles and one edge into \( G_1 \). The free edge is determined by \( \alpha \omega \) to be \( \{x_2, x_4, x_5\} \), since \( x_1 \) is permuted to \( x_4 \) and since \( x_2 \neq x_5 \) guarantees that \( x_2 \) is fixed. Thus, the free edge is part of a triangle \( G[x_2, x_3, x_4] \), as desired. This concludes Subcase IIB.

To complete Case II and the proof of the theorem, we verify that all the hypotheses, and hence the final conclusion, of Lemma 9.5 apply to \( G_1 \) and \( G_2 \), and then we show that \( G \) is of type 1.

Since we have assumed that 
\[ b = b_1 + 1 + b_2 \]
triangles do not embed in \( G \), and since \( b_1 + 1 \) triangles embed in \( G_1 + x_3 = G[X_1 + x_3] \), we know that we cannot embed \( b_2 \) triangles in \( G_2 - x_3 \). Now, 
\[ |V(G_2) - x_3| = 3b_2 + 1, \]
and by (9.22), 
\[ \delta(G_2 - x_3) \geq \delta(G_2) - 1 \]
\[ \geq \frac{2}{3} p_2 - \frac{1}{3} - 1 \]
\[ = \frac{2}{3} (|x_2 - x_3| - 1) \]
\[ = 2b_2. \]
Since
\[ \delta(G) = 2b = 2b_1 + 2 + 2b_2, \]
and since
\[ \text{deg}_{G_2}(v_j) = 2b_2 + 1 \quad (j = 1, 2), \]
each \( v_j \) \((j = 1, 2)\) is adjacent to at least \(2b_1 + 1\) vertices of \( G_1 \). Hence, there are at least
\[ |E(v_1, X_1)| + |E(v_2, X_1)| - p_1 \geq 2(2b_1 + 1) - (3b_1 + 2) = b_1 \geq 1 \]
choices \( y_3 \in X_1 \) such that \( G[v_1, v_2, y_3] \) is a triangle.
Therefore, as we already remarked, we may apply Lemma 9.6 and conclude that both \( G_1 - y_3 \) and \( G_2 - x_3 \) are of type 1.

Next, we establish the hypotheses of Lemma 9.5.

Since \( G_1 - y_3 \) and \( G_2 - v_3 \) are of type 1, where
\[ v_3 = x_3', \]
and since they have \( 3b_1 + 1, 3b_2 + 1 \) vertices, respectively, there are sets \( Y_3' \subseteq X_1 - y_3 \) and \( V_3' \subseteq X_2 - v_3 \) with
\[ |Y_3'| = b_1 - 1, \]
\[ |V_3'| = b_2 - 1, \]
such that \( G_1 - y_3 - Y_3' \) and \( G_2 - v_3 - V_3' \) are complete bipartite graphs \( Y_1 \cup Y_2 \) and \( V_1 \cup V_2 \), respectively. Define
\[ Y_3 = Y_3' + y_3, \]
\[ V_3 = V_3' + v_3. \]
Therefore, by the induction hypothesis, $G_2 - x_3$ is of type 1 or type 2. Hence,
\[
\delta(G_2 - x_3) = \frac{2}{3}(p_2 - 2) = \frac{2}{3}p_2 - \frac{4}{3},
\]
whence, by (9.22), $x_3$ is adjacent to every vertex of $G_2 - x_3$ having degree $\frac{2}{3}p_2 - \frac{4}{3} = \delta(G_2 - x_3)$ in $G_2 - x_3$.

By Lemmas 9.7 and 9.8, we know that $b_2 - 1$ triangles embed in $G_2 - x_3$, and that such an embedding uses all but 4 vertices of $G_2 - x_3$. Moreover, these 4 vertices all have degree $\delta(G_2 - x_3)$ in $G_2 - x_3$, and they induce a quadrilateral. Now, $x_3$ is adjacent to all four of these vertices, and hence forms a triangle with 2 of them.

Let $v_1$ and $v_2$ denote the other 2 vertices on this quadrilateral. Note that $v_1$ and $v_2$ are adjacent. We shall show that there are $b_1$ choices of a vertex $y_3 \in X_1$ such that $G[v_1, v_2, y_3]$ is a triangle. If $b_1$ disjoint triangles can be embedded in $G_1 - y_3$, then, counting the triangle containing $x_3$, the triangle $G[v_1, v_2, y_3]$, and the $b_2 - 1$ triangles of $G_2 - x_3$, we have $b$ pairwise disjoint triangles in $G$, contrary to assumption. Hence, $b_1$ triangles do not embed in $G_1 - y_3$. For this to happen, $b_1 \geq 1$.

Thus, by (9.22), we may apply Lemma 9.6, with
\[
\{z, z'\} = \{x_3, y_3\},
\]
and conclude that both $G_1 - y_3$ and $G_2 - x_3$ are of type 1.
Thus, (9.4) and (9.5) of Lemma 9.5 hold, and also

\[(9.32) \quad |Y_3 \cup V_3| = b_1 + b_2 = b - 1.\]

Since \(G_1 - y_3\) is of type 1, if \(y \in Y_1 \cup Y_2\), then

\[
\deg_{G_1 - y_3}(y) = 2b_1 = \frac{2}{3}(p_1 - 2).
\]

Now, \(\delta(G_1) \geq \frac{2}{3} p_1 - \frac{1}{3}\), and hence \(y\) is adjacent to \(y_3 \in X_1\).

Therefore, for any \(y \in Y_1 \cup Y_2\),

\[
\deg_{G_1}(y) = \frac{2}{3}(p_1 - 2) + 1 = \delta(G_1),
\]

and (9.7) of Lemma 9.5 is established. Similarly, since \(G_2 - v_3\) is of type 1, (9.6) may be established, and also for any \(v \in V_1 \cup V_2\),

\[
\deg_{G_2}(v) = \delta(G_2).
\]

By (9.22),

\[
\delta(G_1) + \delta(G_2) \geq \frac{2}{3} p_1 - \frac{1}{3} + \frac{2}{3} p_2 - \frac{1}{3}
\]

\[
= \frac{2}{3}(p - 1)
\]

\[
= \delta(G),
\]

and (9.3) is established. Thus, having proved (9.3) through (9.7) of Lemma 9.5, we conclude from Lemma 9.5 that any vertex \(y \in Y_1 \cup Y_2\) is adjacent to every vertex in \(V_j\) for some \(j \in \{1,2\}\).
Suppose by way of contradiction that some \( y \in Y_1 \cup Y_2 \) is adjacent in \( G \) to vertices \( v_1 \in V_1 \) and \( v_2 \in V_2 \) (i.e., suppose that (9.8) is false). Thus, \( G[y, v_1, v_2] \) is a triangle. By Lemma 9.7, for any vertices \( v_5 \in V_1 \cap V_2 \) and \( v_4 \in V_2 - v_2 \), there is an embedding of \( b_2 - 1 \) triangles into \( G - \{v_1, v_2, v_3, v_4, v_5\} \), since \( G_2 - v_3 \) is of type 1. Note that \( G[v_3, v_4, v_5] \) is also a triangle. We conclude from Lemma 9.7 that for any vertices \( y_1 \in Y_1 \), \( y_2 \in Y_2 \), there is an embedding of \( b_1 - 1 \) pairwise disjoint triangles into \( G_1 - \{y, y_1, y_2, y_3\} \), since \( G_1 - y_3 \) is of type 1. Including the \( b_2 - 1 \) triangles of \( G_2 - \{v_1, v_2, v_3, v_4, v_5\} \) and the 3 triangles \( G[y_1, y_2, y_3], G[y, v_1, v_2], \) and \( G[v_3, v_4, v_5] \), we have

\[
(b_1 - 1) + (b_2 - 1) + 3 = b
\]

pairwise disjoint triangles embedded in \( G \), contrary to assumption. Hence, (9.8) holds, and by Lemma 9.5, \( G - (Y_3 \cup V_3) \) is a complete bipartite graph. By (9.32) and Lemma 9.3, \( G \) is of type 1. This completes the proof of Theorem 9.2.
10. Subgraphs of graphs, III

We give one of our main results in this section.

**Theorem 10.1** Let $G$ and $H$ be graphs on $p$ vertices. If $\Delta(H) \leq 2$ and if

\[
\Delta(G^c) \leq \frac{p}{3} - \max(k, \frac{3}{2}p^{1/3}),
\]

where $k = 9$, then $H$ is a subgraph of $G$.

**Proof:** The entire chapter is devoted to the proof of this Theorem. First, we introduce notation.

Recall that for a bijection

\[
\pi: V(H) \rightarrow V(G),
\]

if $v \in V(G)$, then

\[
M(v) = \{x: \pi^{-1}(x) \text{ and } \pi^{-1}(v) \text{ are adjacent in } H\}.
\]

We shall use the notation

\[
M(v_1, v_2, \ldots, v_n) = \bigcup_{i=1}^{n} M(v_i),
\]

and

\[
M(A) = \bigcup_{v \in A} M(v),
\]

where the latter union is over the vertices $v \in A$, where $A \subseteq V(G)$.

Successors and predecessors are defined as in section 7, except that we do not permit vertices of $M(v)$ to be successors or predecessors of $v$. Thus, $x \in V(G)$ is a successor of $v \in V(G)$ if $x$ is adjacent in $G$ to each
vertex of $M(v)$. The set of successors of $v$ is denoted $S(v)$. Also, $v$ is a predecessor of $x$ whenever $x$ is a successor of $v$. The set of predecessors of $x$ is denoted $P(x)$.

Suppose that $H$ is an edge-minimal graph for which the theorem is false. If $y \in V(H)$ is a vertex of degree 1, then let $e \in E(H)$ be the incident edge. Otherwise, all vertices of $H$ have degree either 0 or 2. If this is the case, let $y \in V(H)$ be a vertex of degree 2, and let $e \in E(H)$ be an edge incident with $y$. By the edge-minimality of $H$, there is, in either case, an embedding

$$\pi: V(H) \rightarrow V(G)$$

of $H - e$ into $G$. Let $\pi(y)$ be denoted by $x$. The bijection $\pi$ and the vertex $x$ are considered fixed throughout the proof. At a relatively early stage in the proof (prop. 10.5), we shall dispose of the case in which $H$ has a vertex of degree 1.

Henceforth, all vertices denoted in this proof by a letter are vertices of $G$.

We define an alternating chain from $x_0$ to $v$ to be any finite sequence of at least 2 distinct vertices $x_0, x_1, \ldots, x_m$ of $V(G)$, with $x_m = v$, such that

$$\begin{align*}
(10.2) & \quad x_i \in S(x_{i-1}) \quad \text{for } i = 1, 2, \ldots, m; \\
(10.3) & \quad \text{If } x_i = x_j \text{ and } i < j, \text{ then either } x_i = x_{i+1} = \ldots = x_j, \text{ or both } x_0 = x_1 = \ldots = x_i \text{ and } x_j = \ldots = x_m = x_0; \\
\text{and} \quad (10.4) & \quad x_i \notin M(x_j) \quad \text{for } 0 \leq i < j \leq m.
\end{align*}$$
Note that (10.2) is equivalent to \( x_{i-1} \in P(x_1) \), and (10.4) is equivalent to \( x_j \notin M(x_1) \). If \( x_0 = v \) in an alternating chain from \( x_0 \) to \( v \), we say that the chain is closed.

The proof of Theorem 10.1 will rest upon the observation that for a closed alternating chain \( x_0, x_1, \ldots, x_m \), with \( x = x_0 = x_m \), \( (x_0, x_1, \ldots, x_{m-1}) \) \( n \) is an embedding of \( H \) into \( G \).

Define, for each integer \( t \geq 1 \), the set \( D_t(x) \) to be the set of all vertices \( z \) such that for any \( t - 1 \) vertices \( w_1, w_2, \ldots, w_{t-1} \in V(G) - M(x, z) \), there is an alternating chain from \( x \) to \( z \) containing no vertex of \( M(w_1, \ldots, w_{t-1}) \). Define \( D_0(x) = V(G) \). Thus, we have

\[
(10.5) \quad D_0(x) \supseteq D_1(x) \supseteq D_2(x) \supseteq \ldots \supseteq D_t(x) \supseteq \ldots \supseteq S(x).
\]

We first prove 15 propositions. Then we break the proof into six cases, and use the propositions and two key lemmas about alternating chains to show that in each case there is a permutation \( \alpha : V(G) \to V(G) \) such that \( \alpha \) embeds \( H \) into \( G \).

Recall that in section 1 we defined, for \( x_1, x_2, \ldots, x_n \in X \), where \( x_i = x_j \) and \( i < j \) imply \( x_i = x_{i+1} = \ldots = x_j \), the symbol \((x_1, x_2, \ldots, x_n)\)' to be the permutation obtained by suppressing multiple successive occurrences of the same member of \( X \) in \( x_1, x_2, \ldots, x_n \). Thus, if \( v_0, v_1, v_2 \in V(G) \) are distinct and if \( v_2 = v_3 \), then

\[(v_0, v_1, v_2, v_3)' = (v_0, v_1, v_2).\]
Prop. 10.2 The number $|P(v)|$ of predecessors of a vertex $v \in V(G)$ is at least

$$\frac{D}{3} + \max(2k, 3p^{1/3}) - 2.$$  

Proof: A vertex $v'$ is not a predecessor of $v$ if there is a vertex $u \in V(G)$, either equal or adjacent in $G^c$ to $v$, such that $v' \in M(u)$. Since $|M(u)| \leq 2$, (10.1) implies that there are at most

$$\frac{2p}{3} - \max(2k, 3p^{1/3}) + 2$$

non-predecessors of $v$. Prop. 10.2 follows.

Prop. 10.3 The number $|S(v)|$ of successors of a vertex $v$ is at least

$$\frac{D}{3} + \max(2k, 3p^{1/3}) - 2.$$  

Proof: The non-successors of $v$ are the vertices which are either equal or adjacent in $G^c$ to a vertex of $M(v)$. For each $u \in M(v)$, the number of vertices either equal or not adjacent to $u$ is at most

$$\frac{D}{3} - \max(k, \frac{3}{2}p^{1/3}) + 1.$$  

Since $|M(v)| \leq 2$, there are at most

$$\frac{2p}{3} - \max(2k, 3p^{1/3}) + 2$$

non-successors of $v$. Prop. 10.3 follows.

Prop. 10.4 If $z \in D_t(x)$ and $t \geq 1$, then $z \not\in P(x) + x$.

Proof: If Prop. 10.4 is false, then there is a closed alternating chain $x = x_0, x_1, \ldots, z, x$, and so $(x_0 x_1 \ldots z)^\pi$ embeds $H$ into $G$. This is the conclusion of Theorem 10.1, which we have assumed to be false.
Prop. 10.5  Every vertex of \( H \) has degree either 0 or 2. In particular, \(|M(x)| = 2\), and if \(|M(v)| = 0\) for some \( v \in V(G) \), then \( v \in P(x) \).

**Proof:** If \(|M(x)| = 1\), then let \( M(x) = x' \). The successors of \( x \) are the vertices adjacent in \( G \) to \( x' \). Thus, by (10.1), \(|S(x)| > \frac{2D}{3}\), and since, by Prop. 10.2, there are more than \( \frac{D}{3} \) predecessors of \( x \), there is a vertex \( x_1 \in S(x) \setminus P(x) \). Then \( (x, x_1) \) embeds \( H \) into \( G \), contrary to the assumption that \( H \) is a graph for which the theorem is false. Hence, \(|M(x)| \) is not 1, and by the original choice of \( x \), every vertex of \( H \) has degree 0 or 2. The final statement of the proposition follows because, by the definition of successors, if \(|M(v)| = 0\), then \( S(v) = V(G) \).

**Definitions** If \( z^* \in D_1(x) - D_t(x) \), for some \( t \geq 2 \), define \( C(z^*) \) to be the set of all alternating chains from \( x \) to \( z^* \). Since \( z^* \not\in D_t(x) \), then by definition of \( D_t(x) \), there is a set \( A(z^*) = \{w_1, \ldots, w_s\} \subseteq V(G) - M(x, z^*) \), with minimum possible integer \( s \leq t - 1 \), such that every chain in \( C(z^*) \) has a vertex in \( M(A(z^*)) \). Of course, \( A(z^*) \) is not necessarily uniquely determined. However, we shall consider the set \( A(z^*) \) to be fixed, for each \( z^* \in D_1(x) - D_t(x) \). We have
\[ |A(z^*)| \leq t - 1, \]

and we have
\[ |M(A(z^*))| \leq 2t - 2. \]

**Prop. 10.6** Let \( t \geq 2 \) be an integer. If
\[ (10.6) \quad z \in D_t(x) \]

and if
\[ (10.7) \quad z^* \in S(z) - D_t(x), \]
then one of the following three statements holds:
\[ (10.8) \quad z \notin M(A(z^*)) \cap D_{t+1}(x) \text{ and } z^* \in D_{t-1}(x); \]
\[ (10.9) \quad z \in M(A(z^*)); \]
\[ (10.10) \quad z^* \in M(x). \]

If (10.10) is false, then also,
\[ (10.11) \quad z^* \notin P(x) + x. \]

**Proof:** Let \( z, z^* \in V(G) \) satisfy (10.6) and (10.7). First, we claim that either (10.10) holds, or \( C(z^*) \) is not empty. By (10.6) and \( t \geq 2 \), there is a chain \( C \) from \( x \) to \( z \) avoiding \( M(z^*) \). If \( C \) passes through \( z^* \), then \( C(z^*) \) is not empty. Otherwise, we extend the chain \( C \) by adding \( z^* \) at the end and we denote the resulting sequence by \( C^* \). Since \( z^* \in S(z) \), \( C^* \) is an alternating chain, provided some vertex in \( M(z^*) \) does not already occur in \( C^* \). By our choice of \( C \), unless (10.10) holds, this condition is satisfied. This justifies the claim.
Henceforth, we assume that (10.10) is false and thus that \( C(z^*) \) is not empty. Therefore, \( A(z^*) \) exists, and \( z^* \in D_1(x) \), whence (10.11) follows, by Prop. 10.4.

By (10.6) and (10.7),

\[
(10.12) \quad z \in P(z^*) \cap D_t(x).
\]

Suppose by way of contradiction that (10.9) is false and

\[
(10.13) \quad z \in D_{t+1}(x).
\]

Then either there is an alternating chain \( C \) from \( x \) to \( z \) that misses \( M(A(z^*) + z^*) \), or by definition of \( D_{t+1}(x) \), the sets \( A(z^*) + z^* \) and \( M(x, z) \) intersect.

We quickly dispose of the latter possibility.

From (10.12), \( z \in P(z^*) \), and hence, \( z^* \notin M(z) \). Since (10.10) is assumed to be false, \( z^* \notin M(x) \). Since (10.9) is assumed to be false, \( M(z) \) cannot intersect \( A(z^*) \).

Finally, the definition of \( A(z^*) \) assures us that \( M(x) \) and \( A(z^*) \) do not overlap.

Thus, we can assume that there is an alternating chain \( C \) from \( x \) to \( z \) that misses \( M(A(z^*) + z^*) \). If \( z^* \) does not occur in \( C \), let \( C^* \) denote the sequence obtained by appending \( z^* \) to the end of the sequence \( C \). Since \( C \) misses \( M(z^*) \), \( C^* \notin C(z^*) \), unless \( z^* \) occurs in \( C \). But if \( z^* \) occurs in \( C \), let \( C^* \) instead denote the subsequence of \( C \) terminating at \( z^* \). Since \( C \) misses \( M(A(z^*)) \), and since, by definition of \( A(z^*) \), \( z^* \) also misses \( M(A(z^*)) \),
so does \( C^* \in C(z^*) \), contrary to the definition of \( A(z^*) \). This contradiction shows that (10.13) is false, and thus the first part of (10.8) holds.

Next, supposing that (10.9) and (10.10) are false, we shall prove the last part of (10.8). We proceed by induction on \( t \).

As a basis for induction, we note that the fact that \( C(z^*) \) is nonempty implies that \( z^* \in D_1(x) \), by definition of \( D_1(x) \). Therefore, the last part of (10.8) holds when \( t = 2 \).

Suppose that Prop. 10.6 is true for integers less than \( t \), where \( t \geq 3 \). Suppose, contrary to (10.8), that

(10.14) \( z^* \notin D_{t-1}(x) \).

Thus, (10.5) and (10.6) imply

(10.15) \( z \in D_{t-1}(x) \),

and (10.14) and (10.7) imply

(10.16) \( z^* \in S(x) - D_{t-1}(x) \).

Note that (10.15) and (10.16) are simply (10.6) and (10.7) with \( t - 1 \) in place of \( t \). Hence, by the induction hypothesis, since (10.9) and (10.10) are false, we must have \( z \notin D_t(x) \). But this contradicts (10.6) itself. Therefore, (10.8) is proved. This proves Prop. 10.6.
We define a succession to be an ordered pair \((u, v)\) of vertices such that \(v \in S(u)\).

**Prop. 10.7** There is an integer \(t \geq 2\) such that the number of successions \((u, v)\) with \(u \in D_t(x), v \notin D_t(x)\) is at most \(p^{4/3} + \frac{2}{3}p\).

**Proof:** We have the following three upper bounds on different types of successions, for \(t \geq 2\).

(10.17) The number of successions \((u, v)\), with \(u \in D_t(x), v \in M(x)\) is at most \(2|D_t(x)|\);

(10.18) The number of successions \((u, v)\) with \(v \notin D_{t-1}(x) \cup M(x)\) and \(u \in D_t(x) \cap P(v)\), with \(|D_t(x) \cap P(v)| \leq 2t - 2\), is at most \((p - |P(x)| - 1 - |D_{t-1}(x)|)(2t - 2)\);

(10.19) The number of successions \((u, v)\), with \(u \in D_t(x), v \notin D_t(x)\) and \(|D_t(x) \cap P(v)| > 2t - 2\) is at most \((|D_{t-1}(x) - D_t(x)|)(2t - 2 + |D_t(x) - D_{t+1}(x)|)\).

Statement (10.17) holds because by Prop. 10.5, \(|M(x)| = 2\).

We obtain (10.18) by using (10.11) of Prop. 10.6, which asserts that \(v \notin P(x) + x\). Next, we justify the bound of (10.19).
Suppose that for a given vertex \( v \notin D_t(x) \cup M(x) \), there are more than \( 2t - 2 \) successions of the form \( (u,v) \), where \( u \in D_t(x) \). Thus, we have excluded successions counted in (10.17) and (10.18). By Prop. 10.6 with \( u = z \) and \( v = z^* \), there is a set \( M(A(v)) \) of at most \( 2t - 2 \) vertices in \( V(G) \) such that (10.8), (10.9), or (10.10) of Prop. 10.6 holds. We cannot have (10.10), since \( v \in M(x) \) has been excluded. Note that if for each \( u \), condition (10.9) holds, then there are at most \( 2t - 2 \) values of \( u \), another case already excluded. Hence, there is a value of \( u \), say \( u = u_0 \), such that \( u_0 \notin M(A(v)) \). Then we have (10.8), whence \( u_0 \in D_t(x) - D_{t+1}(x) \), and \( v \in D_{t-1}(x) \). Therefore, in general, since

\[
 u \in M(A(v)) \cup (D_t(x) - D_{t+1}(x))
\]

and

\[
 v \in D_{t-1}(x) - D_t(x)
\]

for all successions \( (u,v) \) not counted in (10.17) or (10.18), the bound of (10.19) holds.

We write

\[
(10.20) \quad a_t = |D_t(x) - D_{t+1}(x)|
\]

The total number of successions \( (u,v) \) with \( u \in D_t(x) \), \( v \notin D_t(x) \) is, by (10.17), (10.18), and (10.19), at most
\[(10.21) \quad a_{t-1}(2t-2 + a_t) + (p - |P(x)| - 1 - |D_{t-1}(x)|)(2t-2) + 2|D_t(x)| \]
\[= a_{t-1}a_t + (p - |P(x)| - 1 - |D_t(x)|)(2t-2) + 2|D_t(x)|.\]

Let
\[(10.22) \quad b = p^{4/3}\]
and let
\[(10.23) \quad c = 2p/3.\]

Suppose, by way of contradiction, that for all \(t\) satisfying \(2 \leq t \leq b/c\),
\[(10.24) \quad a_t a_{t-1} \geq b - ct.\]

Since
\[
0 \leq (\sqrt{a_t} - \sqrt{a_{t-1}})^2 = a_t - 2\sqrt{a_t a_{t-1}} + a_{t-1},
\]
we have from this and (10.24),
\[(10.25) \quad \sqrt{(b - ct)} \leq \sqrt{(a_t a_{t-1})} \leq \frac{1}{2}(a_t + a_{t-1}).\]

Summing (10.25) from \(t = 2\) to \(n = \lfloor b/c \rfloor\), we get
\[(10.26) \quad \Sigma_{t=2}^{n} \sqrt{(b - ct)} \leq -\frac{1}{2}a_1 - \frac{1}{2}a_n + \Sigma_{t=2}^{n} a_t < \Sigma_{t=2}^{n} a_t.\]

By the Fundamental Theorem of Calculus, and (10.26),
\[(10.27) \quad \frac{2b^{3/2}}{3c} = \int_0^{b/c} \sqrt{(b - cx)} \, dx \]
\[= \int_0^{2} \sqrt{(b - cx)} \, dx + \int_{b/c}^{b} \sqrt{(b - cx)} \, dx \]
\[\leq \int_0^{2} \sqrt{b} \, dx + \Sigma_{t=2}^{n} \sqrt{(b - ct)} < 2\sqrt{b} + \Sigma_{t=2}^{n} a_t.\]
We combine (10.22), (10.23), (10.27), Prop. 10.4, and Prop. 10.2 to obtain

\[
p = \frac{2b^{3/2}}{3c} < 2p^{2/3} + \sum_{t=2}^{n} a_t
\]

\[
= 2p^{2/3} + |D_2(x) - D_{n+1}(x)|
\]

\[
\leq 2p^{2/3} + p - |P(x) + x|.
\]

\[
< 2p^{2/3} + \frac{2p}{3} - 14,
\]

which is clearly false for all \(p\). Hence, there is a \(t\) such that (10.24) is false.

**Fix \(t\) throughout the rest of the proof** so that

(10.28) \(a_t a_{t-1} < b - ct\).

**Throughout the rest of the proof, let**

(10.29) \(D_t(x) = D(x)\),

for this value of \(t\) satisfying (10.28).

Thus, by (10.28), (10.29), (10.21), (10.5) and Props. 10.4, 10.3, and 10.2, and by (10.22) and (10.23),

\[
b - ct + (p - |P(x)| - 1 - |D(x)|)(2t - 2) + 2|D(x)|
\]

\[
< b - ct + (2t - 2)\frac{p}{3} + 2(\frac{2p}{3})
\]

\[
= p^{4/3} + \frac{2p}{3}.
\]

This proves Prop. 10.7.
Prop. 10.8 For any \( u_0 \in D(x) \), the number of successions of the form \((u,v)\), with \( u,v \in D(x) - u_0 \), is at least
\[
(\vert D(x) \vert - 1) \left( \frac{D}{3} + \max(2k, 3^{1/3} p^{1/3}) - 3 \right) - p^{4/3} - \frac{2}{3} p.
\]

Proof: By Prop. 10.3, the number of successions of the form \((u,v)\), with \( u \in D(x) - u_0 \) and \( v \neq u_0 \) is at least
\[
\vert D(x) - u_0 \vert \left( \frac{D}{3} + \max(2k, 3^{1/3} p^{1/3}) - 2 - \vert u_0 \vert \right).
\]
By Prop. 10.7, at most \( p^{4/3} + \frac{2p}{3} \) of these are not of the form with \( v \in D(x) \). This implies the proposition.

Prop. 10.9 There are distinct vertices \( u_0, v_0 \) in \( D(x) \) such that
\[
(10.30) \quad |P(u_0) \setminus D(x)| \geq |P(v_0) \setminus D(x)| > \frac{D}{3} - 5.
\]

Proof: Let \( u_0 \in D(x) \) be a vertex having the most predecessors in \( D(x) \). Let \( v_0 \) denote a vertex in \( D(x) - u_0 \) having the most predecessors in \( D(x) - u_0 \). Clearly, the first inequality of (10.30) holds. Note that
\[
|P(v_0) \setminus D(x)|
\]
is at least the average number of successions per vertex in \( D(x) - u_0 \), whence, by Prop. 10.8,
\[
(10.31) \quad |P(v_0) \setminus D(x)| \geq \frac{D}{3} + \max(2k, 3^{1/3} p^{1/3}) - 3
- \frac{p^{4/3} + 2p/3}{|D(x)|} - 1
\]
By Prop. 10.3, and since \( S(x) \subseteq D(x) \),
\[(10.32) \quad |D(x)| - 1 \geq |S(x)| - 1 \]
\[\geq \frac{p}{3} + \max(2k, 3p^{1/3}) - 3 \]
\[> \frac{p}{3}.
\]
By \((10.31)\) and \((10.32)\),
\[|P(v_0) \setminus D(x)| > \frac{p}{3} + \max(2k, 3p^{1/3}) - 3 - 3p^{1/3} - 2 \]
\[\geq \frac{p}{3} - 5,
\]
and hence, \((10.30)\) holds.

**Remarks:** Vertices \(u_0\) and \(v_0\) satisfying Prop. 10.9 are chosen, and will remain fixed throughout the rest of the proof of Theorem 10.1.

Also, for the remainder of the proof, we shall use \((10.1)\) and Props. 10.2 and 10.3 in their weaker form, without the term involving \(p^{1/3}\). Thus, the inequalities of \((10.1)\) and Props. 10.1 and 10.2 will be replaced by

\[\Delta(G^c) \leq \frac{p}{3} - k,\]
\[|P(v)| \geq \frac{p}{3} + 2k - 2,\]
\[|S(v)| \geq \frac{p}{3} + 2k - 2,\]
respectively.

**Prop. 10.10** We have both
\[|S(x) \cap P(u_0)| > 4k - 8,\]
and
\[|S(x) \cap P(v_0)| > 4k - 8,\]
where \(u_0\) and \(v_0\) are the fixed vertices of Prop. 10.9.
Proof: Since the proofs are identical for \( u_0 \) and \( v_0 \), we shall only state the proof for \( v_0 \). By Prop. 10.2,

\[
|V(G) - P(x)| \leq \frac{2p}{3} - 2k + 2.
\]

Since, by Prop. 10.4,

\[
D(x) \subseteq V(G) - P(x) - x,
\]

(10.33) gives

\[
|D(x)| \leq \frac{2p}{3} - 2k + 1.
\]

By (10.34), Prop. 10.9, Prop. 10.3, and \( S(x) \subseteq D(x) \), we have

\[
|S(x) - P(v_0) - D(x)| \geq |S(x) - D(x)| + |P(v_0) - D(x)| - |D(x)|
\]

\[
> \left( \frac{p}{3} + 2k - 2 \right) + \left( \frac{p}{3} - 5 \right) - \left( \frac{2p}{3} - 2k + 1 \right)
\]

\[
\geq 4k - 8.
\]

Prop. 10.11 Suppose that \( v_1 \) and \( v_2 \) satisfy

(10.35) \( v_1 \in S(v_0) - M(x) \)

and

(10.36) \( v_2 \in S(v_1) - M(x, v_0) \).

Then \( v_2 \notin P(x) + x \), and either \( v_0 = v_2 \) or

\[
|S(v_2) - (P(x) + x)| \geq \frac{p}{3} + 2k - 8.
\]

A similar statement holds when \( v_0, v_1, v_2 \) are replaced by \( u_0, u_1, u_2 \), respectively.
Proof: By Prop. 10.10, since \( k > 4 \), there is a vertex

\[
x_1 \in S(x) \setminus P(v_0) - M(v_1, v_2) - \{ v_0, v_1, v_2 \},
\]

and \( x_1 \in S(x) \setminus P(v_0) \) guarantees that \( x_1 \notin M(x, v_0) \) and \( v_0 \subseteq S(x_1) \). We claim that \( x, x_1, v_0, v_1, v_2 \) is an alternating chain, or that \( v_0 = v_2 \). To see this, observe first that (10.2) holds for this sequence. Suppose next, that (10.3) fails for this sequence. By the choice of \( x_1, x_1 \notin \{ x, v_0, v_1, v_2 \} \). Thus, for (10.3) to fail, either \( x \notin \{ v_0, v_1 \} \) or \( v_0 = v_2 \). If \( x \in \{ v_0, v_1 \} \), then either \( x, x_1, v_0 \) or \( x, x_1, v_0, v_1 \) is a closed alternating chain, whence \( (x, x_1, v_0)_N \) or \( (x, x_1, v_0, v_1)_N \), respectively, embeds \( H \) into \( G \), contrary to the assumption that \( H \) is not a subgraph of \( G \). If \( v_0 = v_2 \), then the proposition follows immediately, since \( v_0 \notin P(x) + x \). Suppose, finally, that \( x, x_1, v_0, v_1, v_2 \) is not an alternating chain because (10.4) fails. Thus, there are vertices

\[
y_1, y_2 \notin \{ x, x_1, v_0, v_1, v_2 \}
\]
such that \( y_0 \in M(y_2) \). By definition of successors, \( y_1 \) and \( y_2 \) cannot be consecutive vertices of \( x, x_1, v_0, v_1, v_2 \). Since \( v_0 \in D(x) \), we have \( v_0 \notin M(x) \). By the definitions of \( v_1, v_2 \), and \( x_1 \), we exclude \( v_1 \notin M(x), v_2 \notin M(x), v_1 \in M(x_1), v_2 \notin M(x_1) \) and \( v_2 \notin M(v_0) \). Thus, if \( v_0 \neq v_2 \), then (10.2),
(10.3), and (10.4) hold, and so \(x, x_1, v_0, v_1, v_2\) is an alternating chain.

If \(v_2 \in P(x) + x\), then we are done, for 
\[(x x_1 v_0 v_1 v_2)\pi\] would be an embedding of \(H\) into \(G\).

If there exists a vertex 
\[v_3 \in S(v_2) \setminus P(x) - M(v_0, v_1, x_1),\]
then \(x, x_1, v_0, v_1, v_2, v_3, x\) is a closed alternating chain, 
and we are done, for \((x x_1 v_0 v_1 v_2 v_3)\pi\) embeds \(H\) into \(G\). Otherwise, all members of \(S(v_2) \setminus (P(x) + x)\) lie in 
\(M(v_0, v_1, x_1)\), a set of at most 6 members. The number 
of successors of \(v_2\) outside of \(P(x) + x\) is therefore, 
by Prop. 10.3, at least \(\frac{D}{3} + 2k - 2 - 6\). This proves 
Prop. 10.11.

**Prop. 10.12** For any two vertices \(u\) and \(v\) in \(V(G)\), 
the number of predecessors of \(u\) adjacent in \(G\) to \(v\) is 
at least \(3k - 3\).

**Proof:** By Prop. 10.2, 
\[|P(u)| \geq \frac{D}{3} + 2k - 2.\]
Since 
\[\Delta(G^c) \leq \frac{D}{3} - k,\]
at most \(\frac{D}{3} - k + 1\) vertices of \(P(u)\) are adjacent in \(G^c\) 
to \(v\) (one is equal). This leaves at least 
\[(\frac{D}{3} + 2k - 2) - (\frac{D}{3} - k + 1)\]
predecessors of \(u\) adjacent to \(v\).
Prop. 10.13 For any two vertices u and v in V(G), the number of successors of u that are adjacent in G to v is at least $3k - 3$.

Proof: Use the proof of Prop. 10.12, with Prop. 10.2 replaced by Prop. 10.3.

Prop. 10.14 Suppose that $v_1$ satisfies (10.35). Let $y_1, z_2$ be two vertices of G such that $y_1$ is adjacent in G to all successors of $v_1$. Then the number of successors of $v_1$ outside $P(x) + x$ that are adjacent in G to $z_2$ and $y_1$ is at least $3k - 7$.

Proof: By the first conclusion of Prop. 10.11, at most 4 successors $v_2$ of $v_1$ lie in $P(x) + x$ (namely, $M(x, v_0)$), whence, by Prop. 10.3, at least

$$|S(v_1) - P(x) - x| \geq \frac{D}{3} + 2k - 6$$

successors of $v_1$ lie outside $P(x) + x$. At most $\frac{D}{3} - k + 1$ of these are not adjacent in G to $z_2$, by (10.1). All are adjacent to $y_1$, by hypothesis. This leaves at least $3k - 7$ vertices.

Prop. 10.15 If $v_2$ satisfies condition (10.36) of Prop. 10.11 and if $v_2 \neq v_0$, then

$$|S(v_2) \cap P(v_0) - (P(x) + x)| > 4k - 14.$$
Proof: By Prop. 10.11,

\[(10.37) \quad \|S(v_2) - (P(x) + x)\| \geq \frac{p}{3} + 2k - 8.\]

By Prop. 10.4 and 10.9,

\[(10.38) \quad |P(v_0) - (P(x) + x)| \geq |P(v_0) \cap D(x)| \geq \frac{p}{3} - 5.\]

By (10.37), (10.38), and Prop. 10.2,

\[|S(v_2) \cap P(v_0) - (P(x) + x)| \geq |S(v_2) - (P(x) + x)| + |P(v_0) - (P(x) + x)| - |V(G) - (P(x) + x)|\]

\[> \left(\frac{p}{3} + 2k - 8\right) + \left(\frac{p}{3} - 5\right) - p + \frac{p}{3} + 2k - 1\]

\[= 4k - 14.\]

Prop. 10.16 For appropriate vertices \(x_1 \in S(x),\)
\(u_1 \in S(u_0) - M(x),\) \(v_1 \in S(v_0) - M(x),\) and \(z_1 \in V(G),\) one of
the following six cases holds:

\[(10.39) \quad x_1 \in M(v_1);\]
\[(10.40) \quad x_1 \in M(u_1);\]
\[(10.41) \quad v_1 \in M(u_1);\]
\[(10.42) \quad M(z_1) = \{x_1, v_1\};\]
\[(10.43) \quad M(z_1) = \{x_1, u_1\};\]
\[(10.44) \quad M(z_1) = \{u_1, v_1\}.\]

Proof: Let

\[X = (S(x) \cap (S(u_0) - M(x))) \cup (S(x) \cap (S(v_0) - M(x))) \cup ((S(u_0) - M(x)) \cap (S(v_0) - M(x))].\]

and let

\[X' = S(x) \cup S(u_0) \cup S(v_0) - M(x).\]
Note that the definitions of $S(x), X,$ and $X'$ imply that these sets are disjoint from $M(x)$. If (10.39), (10.40), and (10.41) are false, then for any vertex $z \in X$, 

$$M(z) \subseteq V(G) - X'.$$

If also, (10.42), (10.43), and (10.44) are false, then the sets $M(z)$, where $z$ runs over $X$, are disjoint sets of 2 elements contained in $V(G) - X'$. Hence,

$$(10.45) \quad |V(G) - X'| \geq 2|X|.$$ 

By Prop. 10.3, we have 

$$|S(x) - X| + |X| \geq |S(x)|$$

$$\geq \frac{p}{3} + 2k - 2$$

$$|S(u_0) - M(x) - X| + |X| \geq |S(u_0) - M(x)|$$

$$\geq \frac{p}{3} + 2k - 4,$$

$$|S(v_0) - M(x) - X| + |X| \geq |S(v_0) - M(x)|$$

$$\geq \frac{p}{3} + 2k - 4,$$

whence,

$$(10.46) \quad |S(x) - X| + |S(u_0) - M(x) - X| + |S(v_0) - M(x) - X|$$

$$+ 3|X| \geq p + 6k - 10.$$ 

We also have

$$(10.47) \quad |S(x) - X| + |S(u_0) - M(x) - X| + |S(v_0) - M(x) - X|$$

$$+ |X| = |X'|.$$ 

We combine (10.46) and (10.47) to obtain

$$p + 6k - 10 \leq |X'| + 2|X|,$$

whence

$$p - |X'| + 6k - 10 \leq 2|X|,$$
and so, since $6k > 10$,

$$|V(G) - X'| < 2|X|,$$

in contradiction with (10.45). Prop. 10.16 follows.

In the two lemmas below, we define for vertices of $G$,

$$X = \{x_0, x_1, \ldots, x_n\};$$

$$V = \{v_0, v_1, \ldots, v_m\};$$

$$\alpha = (x_0, x_1, \ldots, x_n);$$

$$\beta = (v_0, v_1, \ldots, v_m).$$

Let $G + \{x_i, v_j\}$ denote the graph obtained from $G$ by
adding to $E(G)$ the edge $\{x_i, v_j\}$, where $x_i, v_j \in V(G)$.

**Lemma 10.17** Let $x_0, x_1, \ldots, x_n, x_0$ be a closed alternating chain in $G + \{x_1, v_1\}$, and let $v_0, v_1, \ldots, v_m, v_0$
be a closed alternating chain in $G + \{x_1, v_2\}$. If

(10.48) $x_1 \in M(v_1),$

(10.49) $x \in X,$

(10.50) $\{x_2, v_2\} \in E(G),$

and if

(10.51) $V \cap (M(X) \cup X) = v_1,$

then $\beta \alpha \pi$ is an embedding of $H$ into $G$.

**Proof:** By (10.48), there is an edge $e'$ of $H$ mapped
by $\pi$ to $v_1, x_1$. Recall that $e$ is the edge of $H$ not mapped
into $E(G)$ by $\pi$ and that $x \in \pi(e)$.

By hypothesis and by (10.49), $\alpha \pi$ embeds $H$ into
$G + \{x_2, v_1\}$ and maps $e'$ to $\{x_2, v_1\}$. Also, by hypothesis,
$\beta\pi$ embeds $H-e$ into $G+\{x_1,v_2\}$, and maps $e'$ to $\{x_1,v_2\}$. By (10.51), $e'$ is the only edge of $H$ affected by both $\alpha$ and $\beta$. Hence, $\beta\pi\alpha\pi$ embeds $H-e'$ into $G$, and since

$$\beta\pi\alpha\pi(e) = \{x_2,v_2\} \in E(G),$$

by (10.50), $\beta\pi\alpha\pi$ embeds $H$ into $G$.

**Lemma 10.18** Let $x_0,x_1,\ldots,x_n,x_0$ be a closed alternating chain in $G+\{x_2,z_1\}$, and let $v_0,v_1,\ldots,v_m,v_0$ be a closed alternating chain in $G+\{v_2,z_1\}$. Also, let $Z = \{z_1,z_2\}$, and let $\gamma = (z_1,z_2)$. If

$$M(z_1) = \{v_1,x_1\}, \quad (10.52)$$

$$x \in X, \quad (10.53)$$

$$\{v_2,z_2\}, \{x_2,z_2\} \in E(G), \quad (10.54)$$

$$(V \cup Z) \cap (M(X) \cup X) = z_1, \quad (10.55)$$

$$(X \cup Z) \cap (M(V) \cup V) = z_1, \quad (10.56)$$

and if

$$Z_2 \in P(z_1), \quad (10.57)$$

then $\gamma\beta\pi\alpha\pi$ embeds $H$ into $G$.

**Proof:** By (10.52), there are edges $e_1,e_2$, respectively, mapped by $\pi$ to $\{x_1,z_1\}$ and $\{z_1,v_1\}$. Recall that $e$ is the only edge of $H$ not mapped into $E(G)$ by $\pi$, and that $x \in \pi(e)$.

By the first hypothesis, and by (10.53), $\alpha\pi\beta\pi$ embeds $H$ into $G+\{x_2,z_1\}$, with $e_1$ mapped to $\{x_2,z_1\}$. Also,
by hypothesis, $\beta\alpha$ embeds $H - e$ into $G + \{v_2, z_1\}$. By
(10.55) and (10.56), no edge of $H$ is affected by both $\alpha$ and $\beta$. Therefore, $\beta\alpha\pi$ embeds $H$ into $G + \{x_2, z_1\} + \{v_2, z_1\}$, with $e_1, e_2$ mapped to $\{x_2, z_1\}, \{v_2, z_1\}$, respectively. By
(10.55) and (10.56), $e_1$, respectively $e_2$, is the only edge affected by both $\alpha$ and $\gamma$, respectively $\beta$ and $\gamma$. Hence, (10.57) ensures that $\gamma\beta\alpha\pi$ embeds $H - \{e_1, e_2\}$ into $G$, and since $\gamma\beta\alpha\pi$ maps $e_1$ and $e_2$ to $\{x_2, z_2\}$ and $\{v_2, z_2\}$, (10.54) ensures that $\gamma\beta\alpha\pi$ embeds $H$ into $G$.

Remark: In the six cases of Prop. 10.16, which we consider below, we shall verify the hypotheses of either Lemma 10.17 or Lemma 10.18, whence we conclude that $H$ is a subgraph of $G$. We shall construct the desired alternating chains one vertex at a time. Each time another vertex is chosen, we take care to ensure that (10.51) or both (10.55) and (10.56) hold, although we shall not say so explicitly. As chains are constructed, we select vertices which satisfy (10.2), (10.3), and (10.4). Again, we do not refer to these three conditions explicitly. The other conditions of the lemmas will be verified explicitly in each of the six cases.
Suppose (10.39) holds. We shall apply Lemma 10.17 to show that \((v_0, v_1, v_2, v_3)\)'s \((x_1, x_2, x_2)\) embeds \(H\) into \(G\), for vertices \(v_2, v_3, x_2\) defined below. We must verify the hypotheses of the lemma. If \(x_1 \in M(v_0)\), then we pick \(v_1\) to equal \(v_0\). This is in compliance with (10.39).

Already by (10.39), we have (10.48), and clearly we have (10.49). By Prop. 10.5, there exists \(w_1 \in V(G)\) such that

\[
M(x_1) = \{v_1, w_1\}.
\]

Define the set

\[
T_1 = \{v_0, v_1, w_1, x, x_1\} \cup M(v_0, v_1, x, x_1).
\]

Since \(x_1, v_1, \) and \(w_1\) are counted twice,

\[
|T_1| \leq 10.
\]

By Prop. 10.12, since

\[
3k - 3 > 10 \geq |T_1|,
\]

a vertex \(x_2 \in P(x) - T_1\) exists adjacent in \(G\) to \(w_1\). Thus \(x_2\) is a successor of \(x_1\) in \(G + \{x_2, v_1\}\), whence \(x, x_1, x_2, x\) is the desired alternating closed chain in \(G + \{x_2, v_1\}\).

Let

\[
T_2 = T_1 \cup (M(x_2) + x_2).
\]

Hence,

\[
|T_2| \leq 13.
\]

By Prop. 10.13, since

\[
3k - 3 > 13 \geq |T_2|,
\]

a vertex \(v_2 \in S(v_1) - T_2\) exists adjacent in \(G\) to \(x_2\), in
according to (10.50). Let

\[ T_3 = T_2 \circ (M(v_2) + v_2). \]

Thus,

\[ |T_3| \leq 16. \]

If \( v_2 \in P(v_0) + v_0 \), then let \( v_3 = v_2 \). Otherwise, by Prop. 10.15, since

\[ 4k - 14 > 16 \geq |T_3|, \]

there is a vertex

\[ v_3 \in S(v_2) \setminus P(v_0) - (P(x) + x) - T_3. \]

Thus, \( v_0, v_1, v_2, v_3, v_0 \) is a closed alternating chain in \( G = G + \{v_2, x_1\} \), as desired. We have chosen the vertices \( v_2, v_3, x_2 \) so that (10.51) holds, whence Lemma 10.17 may be applied.

**Suppose (10.40) holds.** This case proceeds as does the previous case with (10.39), but with \( v \) replaced by \( u \), and so we omit the proof.

**Suppose (10.41) holds.** We shall apply Lemma 10.17 to show that \( (v_0, v_1, v_2, v_3) \) embeds \( H \) into \( G \), for vertices \( v_2, v_3, x_2, x_3, x_4 \) defined below. We must verify the hypotheses of the lemma. Let

\[ x_3 = x. \]

Thus, (10.49) holds. We shall apply Lemma 10.17 with \( u_1 \) corresponding to \( x_1 \) of the lemma, whence by (10.41), (10.48) holds. If \( v_1 \in M(u_0) \), then let \( u_1 = u_0 \) in this argument. Define the set

\[ T_1 = \{u_0, v_0, x\} \sim M(u_0, u_1, v_0, v_1, x). \]
By (10.41), \( \{u_1, v_1\} \in T_1 \), and so by Prop. 10.10, since
\[ 4k - 8 \geq 13 \geq |T_1|, \]
there is a vertex \( x_4 \in S(x) \cap P(u_0) - T_1 \). By Prop. 10.5, there exists \( w_1 \in V(G) \) such that
\[ M(u_1) = \{v_1, w_1\}. \]
Define
\[ T_2 = T_1 \cup (M(x_4) + x_4). \]
By Prop. 10.12, since
\[ 3k - 3 \geq 16 \geq |T_2|, \]
there is a vertex \( x_2 \in P(x) - T_2 \) adjacent in \( G \) to \( w_1 \).
Observe that \( u_0, u_1, x_2, x_3, x_4, u_0 \) is a closed alternating chain in \( G + \{v_1, x_2\} \). Thus, we have the first desired chain of Lemma 10.17. Define
\[ T_3 = T_2 \cup (M(x_2) + x_2). \]
By Prop. 10.13, since
\[ 3k - 3 \geq 19 \geq |T_3|, \]
a vertex \( v_2 \in S(v_1) - T_3 \) exists adjacent in \( G \) to \( x_2 \), in compliance with (10.50). If \( v_2 \in P(v_0) + v_0 \), then let \( v_3 = v_2 \). Otherwise, by Prop. 10.15, since
\[ 4k - 14 \geq 19 \geq |T_3|, \]
a vertex
\[ v_3 \in S(v_2) \cap P(v_0) - T_3 - (P(x) + x) \]
exists. Since \( v_3 \in S(v_2) \), we have \( v_3 \notin M(v_2) \). Thus, \( v_0, v_1, v_2, v_3, v_0 \) is a closed alternating chain in \( G + \{x_1, v_2\} \) that satisfies the conditions of the first chain lemma.
By the lemma, \( H \) is a subgraph of \( G \).
Suppose (10.42) holds. We shall apply Lemma 10.18 to show that \((z_1 z_2)(v_0 v_1 v_2 v_3)^\nu (x_1 x_2)^\mu\) embeds \(H\) into \(G\). Thus, we already have (10.52) and (10.53).

Without loss of generality, we may assume in this case that \(x_1 \notin M(v_0)\), for otherwise, we would use the argument associated with (10.39). By Prop. 10.5, and be (10.42), there is a vertex \(w_1 \in V(G)\) such that

\[ M(x_1) = \{w_1, z_1\}. \]

If the vertex of \(H\) mapped to \(x_1\) lies in a triangular component of \(H\), then \(w_1 = v_1\) and the triangle is mapped to the vertices \(x_1, v_1, z_1\), and hence (10.39) holds.

Hence, without loss of generality, we may assume that \(w_1, x_1, z_1, v_1\) are distinct vertices in the image of a path of \(H\). Thus, for the vertices of \(V, X,\) and \(Z\) already selected, (10.55) and (10.56) hold. Define

\[ T_1 = \{v_0, w_1, x_1\} \cup M(v_0, v_1, x, z_1). \]

Note that

\[ v_1, x_1, z_1 \in M(v_1, z_1) \subseteq T_1, \]

and that

\[ |T_1| \leq 11. \]

By Prop. 10.12, since

\[ 3k - 3 > 11 \geq |T_1|, \]

a vertex \(x_2 \in P(x) - T_1\) exists adjacent in \(G\) to \(w_1\). Thus,
$x, x_1, x_2, x$ is a closed alternating chain in $G + \{x_2, z_1\}$, in compliance with Lemma 10.18. Define

$$T_2 = T_1 \cup (M(x_2) + x_2).$$

By Prop. 10.12, since

$$3k - 3 > 14 \geq |T_2|,$$

a vertex $z_2 \in P(z_1) - T_2$ exists adjacent in $G$ to $x_2$, in accordance with (10.57) and the second part of (10.54).

Define

$$T_3 = T_2 \cup (M(z_2) + z_2).$$

By Prop. 10.5 and (10.42), there exists $y_1 \in V(G)$ such that

$$M(v_1) = \{v_1, z_1\}.$$

Since $y_1 \in M(v_1)$, $y_1$ is adjacent in $G$ to all successors of $v_1 \in S(v_0) - M(x)$. By Prop. 10.14, since

$$3k - 7 > 17 \geq |T_3|,$$

a vertex $v_2 \notin (P(x) + x) \cup T_3$ exists adjacent to $y_1$ and $z_2$.

Hence, (10.54) holds. Define

$$T_4 = T_3 \cup (M(v_2) + v_2).$$

If $v_2 \notin P(v_0) + v_0$, then let $v_3 = v_2$. Otherwise, by Prop. 10.15, since

$$4k - 14 > 18 \geq |T_4|,$$

a vertex $v_3 \in S(v_2) \cap P(v_0) - T_4 - (P(x) + x)$ exists. Thus, $v_0, v_1, v_2, v_3, v_0$ is a closed alternating chain in $G + \{v_2, z_1\}$, as required by Lemma 10.18. Since we have verified the requirements of the lemma, $H$ is a subgraph of $G$. 
Suppose (10.43) holds. This case proceeds just like the proceeding case, except that \( u \) is substituted for \( v \). Thus, we omit the details.

Suppose (10.44) holds. We apply Lemma 10.18 to show that \((z_1, z_2)(v_0, v_1, v_2, v_3, v_4, u_0, u_1, x_2, x_3, x_4)^n\) embeds \( H \) into \( G \), for vertices \( v_2, v_3, x_2, x_3, x_4, z_2 \) defined below. Let 
\[ x_3 = x. \]
Thus, we have (10.53), and by (10.44) with \( u_1 \) equal to \( x_1 \) of Lemma 10.18, we have (10.52). If \( v_1 \in M(u_0) \), then let \( u_1 = u_0 \) in this argument. Let 
\[ T_1 = (u_0, v_0, x_1 \cup M(u_0, u_1, v_0, v_1, x, z_1). \]
Note that by (10.44), 
\[ \{u_1, v_1\} = M(z_1) \subset T_1, \]
and that since \( z_1 \in M(u_1) \setminus M(v_1) \) is twice counted, 
\[ |T_1| \leq 14. \]
By Prop. 10.10, since 
\[ 4k - 8 \geq 14 \geq |T_1|, \]
a vertex \( x_4 \in S(x_1 \setminus P(u_0) \setminus T_1 \) exists. By Prop. 10.5 and (10.44), there exists a vertex \( w_1 \in V(G) \) such that 
\[ M(u_1) = \{z_1, w_1\}. \]
Let 
\[ T_2 = T_1 \cup (M(x_4) + x_4). \]
If the vertex of $H$ mapped to $z_1$ lies in a triangular component of $H$, then $v_1 = w_1$, and the triangle is embedded onto $z_1, u_1, v_1$, and (10.41) holds. Hence, without loss of generality, we may assume that $w_1, u_1, z_1, v_1$ are distinct successive vertices in the image of a path of $H$. Thus, for the vertices already selected, (10.55) and (10.56) hold.

By Prop. 10.12, since
\[ 3k - 3 > 17 \geq |T_2|, \]
a vertex $x_2 \in P(x) - T_2$ exists adjacent in $G$ to $w_1$.

Observe that $u_0, u_1, x_2, x_3, x_4, u_0$ is thus a closed alternating chain in $G + \{x_2, z_1\}$. Thus, we have the first of the alternating chains of Lemma 10.18. Define
\[ T_3 = T_2 \cup (M(x_2) + x_2). \]

By Prop. 10.12, since
\[ 3k - 3 > 20 \geq |T_3|, \]
a vertex $z_2 \in P(z_1) - T_3$ exists adjacent in $G$ to $x_2$.

Hence, the second part of (10.54) holds, and (10.57) holds. By Prop. 10.5, and by (10.44), there is a vertex $y_1 \in V(G)$ such that
\[ M(v_1) = \{y_1, z_1\}. \]

Since $y_1 \in M(v_1)$, $y_1$ is adjacent in $G$ to all successors of $v_1 \in S(v_0) - M(x)$. Let
\[ T_4 = T_3 \cup M(z_2). \]
By Prop. 10.14, since

$$3k - 7 > 19 \geq |T_4 - \{v_0, x, y_1\}|,$$

a vertex

$$v_2 \notin (P(x) + x) \cup (T_4 - \{v_0, x, y_1\})$$

exists adjacent in $G$ to $y_1$ and $z_2$. This verifies (10.54). If $v_2 \in P(v_0) + v_0$, then let $v_3 = v_2$. Otherwise, since

$$4k - 14 \geq 19 \geq |T_4 - (M(v_0) + v_0)|,$$

Prop. 10.15 implies that there is a vertex

$$v_3 \in S(v_2) \cap P(v_0) - T_4 - (P(x) + x).$$

Thus, $v_0, v_1, v_2, v_3, v_0$ is a closed alternating chain in $G + \{v_2, z_1\}$. The other conditions of Lemma 10.18 may be readily verified. Thus, $H$ is a subgraph of $G$.

This completes the proof of Theorem 10.1.
REFERENCES


