Double Cycle Covers and the Petersen Graph

Paul A. Catlin

DEPARTMENT OF MATHEMATICS
WAYNE STATE UNIVERSITY
DETROIT, MICHIGAN

ABSTRACT

Let \( O(G) \) denote the set of odd-degree vertices of a graph \( G \). Let \( t \in \mathbb{N} \) and let \( \mathcal{G}_t \) denote the family of graphs \( G \) whose edge set has a partition

\[
E(G) = E_1 \cup E_2 \cup \cdots \cup E_t,
\]
such that \( O(G) = O(G[E_i]) \ (1 \leq i \leq t) \). This partition is associated with a double cycle cover of \( G \). We show that if a graph \( G \) is at most 5 edges short of being 4-edge-connected, then exactly one of these holds: \( G \in \mathcal{G}_3 \), \( G \) has at least one cut-edge, or \( G \) is contractible to the Petersen graph. We also improve a sufficient condition of Jaeger for \( G \in \mathcal{G}_{2p+1} \ (p \in \mathbb{N}) \).

1. INTRODUCTION

A **double cycle cover** of a graph \( G \) is a collection of cycles of \( G \) (multiplicities allowed) such that each edge of \( G \) is in exactly two cycles of the collection. The family of graphs with double cycle covers is closed under contraction, and a graph with a double cycle cover possesses no cut edge. In [6] and [7] a general reduction method was presented that can be applied in certain situations to determine whether a graph belongs to a given family of graphs that is closed under contraction. We use that reduction method to show that certain graphs have a particular type of double cycle cover.

Szekeres [21] and Seymour [20] conjectured that every graph with no cut edge has a double cycle cover. This is trivial for planar graphs: the collection of facial cycles forms a double cover of a planar graph. Jaeger [14] has written a survey article on this problem, and he notes that it is sufficient to prove the conjecture for 3-regular graphs. Goddyn [10] has shown that if a counterexample exists, then a smallest 3-regular counterexample has girth at least seven. Tarsi [22] proved the conjecture for graphs with a hamilton path. Alspach and Zhang [1] have recently proved this conjecture for 2-connected 3-regular graphs containing no subdivision of the Petersen graph (compare with Conjecture 1 below).
We use the terminology of Bondy and Murty [3], except that a graph is presumed to have no loops, and we regard $K_1$ as having infinite edge-connectivity. A contraction of $G$ is a graph $G'$ obtained from $G$ by contracting a set (possibly empty) of edges and deleting all resulting loops. If $H$ is a connected subgraph of $G$, then $G/H$ denotes the graph obtained from $G$ by contracting the edges of $E(G/[V(H)])$.

An elementary homomorphism of a graph $G$ is a graph $G'$ obtained from $G$ by identifying two vertices in the same component and by deleting any resulting loops. Note: this is not the usual definition of homomorphisms. A homomorphism of $G$ is a graph obtained from $G$ by a sequence (possibly empty) of elementary homomorphisms.

Let $\mathcal{F}$ be a family of graphs (also called a family). We say that $\mathcal{F}$ is closed under contraction (respectively, closed under homomorphisms) if for any $G \in \mathcal{F}$, every contraction (respectively, homomorphism) of $G$ is in $\mathcal{F}$. Let $\mathcal{F}^R$ denote the family of graphs having no nontrivial connected subgraph in $\mathcal{F}$. Any graph in $\mathcal{F}^R$ is called $\mathcal{F}$-reduced. For a family $\mathcal{F}$ and a graph $G$, the graph $G_0$ is called an $\mathcal{F}$-reduction of $G$ if $G_0 \in \mathcal{F}^R$ and $G_0$ is obtained from $G$ by a sequence of contractions of subgraphs in $\mathcal{F}$. For example, if $\mathcal{F} = \{2\text{-edge-connected graphs}\}$ then $\mathcal{F}^R = \{\text{forests}\}$. Reductions are used, with (2) of Theorem 2, to determine which graphs lie in a given family.

**Theorem 1** [7]. If a family $\mathcal{F}$ is closed under homomorphisms, then every graph $G$ has a unique $\mathcal{F}$-reduction.

If a family $\mathcal{F}$ is closed under homomorphisms and if $G$ is a graph, then let $G/\mathcal{F}$ denote the unique $\mathcal{F}$-reduction of $G$.

For a family $\mathcal{F}$, we define the kernel of $\mathcal{F}$ to be the family of connected graphs

$$\mathcal{F}^0 = \{H: \text{For every supergraph } G \text{ of } H, G \in \mathcal{F} \iff G/H \in \mathcal{F}\}. \quad (1)$$

**2. PRIOR RESULTS ON KERNELS**

The kernel of a family is often just $\{K_1\}$, an uninteresting case in which the reduction method says nothing. However, in this paper we will consider certain families with large kernels. The basic reduction theorem is this:

**Theorem 2** [7]. Let $\mathcal{F}$ and $\mathcal{F}$ be families such that $\mathcal{F}$ is closed under contraction and

$$\mathcal{F} \subseteq \mathcal{F}^0.$$

Let $G$ be a graph and let $G'$ be a $\mathcal{F}$-reduction of $G$. Then

$$G' \in \mathcal{F} \iff G \in \mathcal{F}. \quad (2)$$

If $\mathcal{F}$ is also closed under homomorphisms, then the $\mathcal{F}$-reduction of $G$ is unique.
Of course, (2) is a straightforward consequence of the definition of the kernel \( \mathcal{F}^0 \) and the fact that \( \mathcal{T} \subseteq \mathcal{F}^0 \). Only the last part of Theorem 2, which follows from Theorem 1, is nontrivial. The point of Theorem 2 is to simplify problems of characterizing graphs in \( \mathcal{F} \) by restricting the problem to graphs in \( \mathcal{F}_R \).

Define a family \( \mathcal{C} \) of connected graphs to be complete if \( \mathcal{C} \) satisfies these three axioms:

(C1) \( K_1 \in \mathcal{C} \);
(C2) \( \mathcal{C} \) is closed under contraction; and
(C3) \( H \subseteq G, H \in \mathcal{C}, G/H \in \mathcal{C} \Rightarrow G \in \mathcal{C} \).

Proofs of the following theorems may be found in the indicated sources. Theorems 4 and 5 are quite easy.

**Theorem 3** [6]. For any family \( \mathcal{F} \) of graphs, closed under contraction, these are equivalent:

(a) \( \mathcal{F} \) is a complete family,
(b) \( \mathcal{F} = \mathcal{F}^0 \), and
(c) \( \mathcal{F} \) is the kernel of some family closed under contraction.  

**Theorem 4** [6]. For any family \( \mathcal{F} \), \( \mathcal{F}^0 \subseteq \mathcal{F} \Leftrightarrow K_1 \in \mathcal{F} \).  

**Theorem 5** [6]. For any family \( \mathcal{F} \), \( \mathcal{F} \cap \mathcal{F}_R \subseteq \{K_1\} \).  

**Theorem 6** [7]. Any complete family is closed under homomorphisms.  

3. PRIOR RESULTS ON EDGE-CONNECTIVITY

Let \( \tau(G) \) be the maximum number of edge-disjoint spanning trees in \( G \). The following theorem improves a result of Kundu [15] and Gusfield [11]. For \( k = 2 \), Zhan [24] proves the “\( \Rightarrow \)” case of this result:

**Theorem 7** [6]. Let \( G \) be a nontrivial graph, let \( k \) be an integer at most \( |E(G)| \), and let \( \mathcal{C}_k \) be the collection of \( k \)-element subsets of \( E(G) \). Then

\[
\kappa'(G) \geq 2k \Leftrightarrow \forall E' \in \mathcal{C}_k, \tau(G - E') \geq k. 
\]

Let \( uv \) and \( vw \) be two edges of a graph \( G \) with \( u \neq w \). Define the graph \( G(uv, vw) \) to be obtained from \( G - \{uv, vw\} \) by adding a new edge \( uw \). We say that \( \{uv, vw\} \) has been lifted to change \( G \) into \( G(uv, vw) \).

Let \( uv \) and \( vw \) be two edges of a graph \( G \), and suppose that \( d(v) = 2 \). If \( v \) and the two incident edges \( \{uv, vw\} \) are deleted from \( G \) and replaced by the new edge \( uw \), then we say that \( v \) has been dissolved.

For the vertices \( x, y \in V(G) \), let \( \kappa'_G(x, y) \) be the minimum cardinality of a set \( E \subseteq E(G) \) such that \( x \) and \( y \) are in different components of \( G - E \).
Theorem 8 (Mader’s Lifting Theorem [17]). Let \( G \) be a graph and suppose \( v \in V(G) \) has \( d(v) \geq 4 \), is not a cutvertex, and has at least two neighbors. Then there are edges \( e', e'' \) incident with \( v \) such that \( G' = G(e', e'') \) satisfies
\[
\kappa'_{G'}(x, y) = \kappa'_G(x, y) \quad \forall \ x, y \in V(G) - v.
\]

4. EXAMPLES

The following examples will be used throughout the paper. For a family \( \mathcal{F} \) and an integer \( k \in \mathbb{N} \cup \{0\} \), we say that a graph \( G \) is at most \( k \) edges short of being in \( \mathcal{F} \) if \( \mathcal{F} \) has a graph \( G' \) such that \( G \) can be obtained from \( G' \) by deleting at most \( k \) edges.

Let \( t \in \mathbb{N} \). Denote
\[
\mathcal{F}_{t,0} = \{G: \tau(G) \geq t\}
\]
and
\[
\mathcal{F}(t,0) = \{G: \kappa'(G) \geq t\}.
\]

For \( k, t \in \mathbb{N} \), denote
\[
\mathcal{F}_{t,k} = \{G: G \text{ is at most } k \text{ edges short of being in } \mathcal{F}_{t,0}\};
\]
\[
\mathcal{F}(t,k) = \{G: G \text{ is at most } k \text{ edges short of being in } \mathcal{F}(t,0)\}.
\]

We proved [6] that \( \mathcal{F}_{t,0} \) equals the kernel \( \mathcal{F}_{t,k}^0 \) of \( \mathcal{F}_{t,k} \), for any \( k \) and \( t \), and that \( \mathcal{F}^0(t,k) = \mathcal{F}(t,0) \). Hong-Jian Lai [17] proved for any complete family \( \mathcal{C} \), if \( \mathcal{C}(k) \) is the family of graphs \( k \) edges short of being in \( \mathcal{C} \), then \( \mathcal{C}^0(k) = \mathcal{C} \).

For any graph \( H \), let \( O(H) \) denote the set of odd-degree vertices of \( H \). Let \( G \) be a graph and let \( V \subseteq V(G) \). We define a \( V \)-join to be a subset \( E \subseteq E(G) \) such that
\[
O(G[E]) = V.
\]

Define \( \mathcal{FL} \) to be the family of graphs having a spanning closed trail. We call \( \mathcal{FL} \) the family of supereulerian graphs. We regard \( K_1 \in \mathcal{FL} \). Define \( \mathcal{CL} \) to be the family of graphs \( G \) such that for any even subset \( V \subseteq V(G) \), \( G \) has a V-join \( E \) such that \( G - E \) is connected. In [4] we proved that \( \mathcal{CL} \) is contained in \( \mathcal{FL}^0 \), the kernel of \( \mathcal{FL} \), and we proved

Lemma 1. The family \( \mathcal{CL} \) is complete. \( \blacksquare \)
Theorem 9 [4]. If $H$ is a nontrivial subgraph of a $\mathcal{CL}$-reduced graph $G$, then
\[ |E(H)| \leq 2|V(H)| - 3, \]
with equality if and only if $H = K_2$. Also, $\mathcal{S}_{2,0} \subseteq \mathcal{CL}$.  

Theorem 9 is part (iv) of Theorem 8 of [4].

For $t \in \mathbb{N}$, let $\mathcal{S}_t$ denote the family of graphs $G$ whose edge set has a partition
\[ E(G) = E_1 \cup E_2 \cup \cdots \cup E_t, \]
such that each $E_i$ $(1 \leq i \leq t)$ is an $O(G)$-join of $G$. By Theorems 7, 9, and 4, and by a result in [8] that $\mathcal{CL} \subseteq \mathcal{S}_3$, we have
\[ \mathcal{S}(4,0) \subseteq \mathcal{S}_{2,0} \subseteq \mathcal{CL} \subseteq \mathcal{S}_0 \subseteq \mathcal{CL} \subseteq \mathcal{S}_3. \]  

**Lemma 2.** The families $\mathcal{S}_t$, $\mathcal{S}(t,k)$ and $\mathcal{S}_{t,k}$ are closed under contraction.

**Proof.** This follows routinely from the respective definitions.  

To obtain our main result, we shall improve the following result:

Theorem 10 (Jaeger [11]). For all $p \in \mathbb{N}$, $\mathcal{S}_{2p,0} \subseteq \mathcal{S}_{2p+1}$, i.e., if a graph $G$ has $2p$ edge-disjoint spanning trees, then $G \in \mathcal{S}_{2p+1}$.

For the graph $H$, if $O(H) = \emptyset$, then $H$ is called an even graph. For $t \in \mathbb{N}$, let $E_1 \cup E_2 \cup \cdots \cup E_t$ be a partition of $E(G)$ into $O(G)$-joins, and define $E_{i+1} = E_i$. Then for $i \in \{1,2,\ldots,t\}$, $G[E_i \cup E_{i+1}]$ is an even subgraph of $G$. Thus, if $t = 3$ (i.e., if $G \in \mathcal{S}_3$) then three $O(G)$-joins $\{E_1, E_2, E_3\}$ induce a double cycle cover $G[E_1 \cup E_2], G[E_1 \cup E_3], G[E_2 \cup E_3]$ of $G$, consisting of three even subgraphs. Conversely, a double cycle cover of $G$ consisting of three even subgraphs induces a partition of $E(G)$ into three $O(G)$-joins. Our main result is a condition sufficient for $G \in \mathcal{S}_3$.

Tutte [23] conjectured that if a 3-regular graph $G$ is not in $\mathcal{S}_3$ (i.e., if $\chi'(G) = 4$ and $G$ is 3-regular), then $G$ has a subgraph contractible to the Petersen graph. Matthews [18] extended this conjecture by dropping the hypothesis of regularity:

**Conjecture 1** [18,23]. If $G \notin \mathcal{S}_3$ then $G$ has a cut-edge or $G$ has a subgraph contractible to the Petersen graph.

Theorem 14 confirms Conjecture 1 whenever $G \in \mathcal{S}(4,5)$.

Celmins [9] and Preismann [19] conjectured that any graph $G$ with no cut-edge contains a collection of at most five even subgraphs (multiplicities allowed) such that each edge of $G$ is in exactly two of the even subgraphs. Since
an even graph is an edge-disjoint union of cycles (Euler’s Theorem), this is a refinement of the double cycle cover conjecture.

5. THE KERNEL OF $\mathcal{F}_{2p+1}$

Before proving the main result, we need to find a large subfamily of the kernel $\mathcal{F}^O_3$ of $\mathcal{F}_3$. In the process, we shall also improve Theorem 10, due to Jaeger [13], which gives a sufficient condition for $G \in \mathcal{F}_{2p+1}$, where $p \in \mathbb{N}$.

Let $k \in \mathbb{N}$. Denote by $\mathcal{C}_{k+1}$ the family of graphs $H$ with the property that for any $k$ even subsets $V_1, V_2, \ldots, V_k \subseteq V(H)$, there are disjoint sets $E_1, E_2, \ldots, E_k \subseteq E(H)$ such that $O(H[E_i]) = V_i (i = 1, 2, \ldots, k)$, i.e., such that $E_i$ is a $V_i$-join in $H$. The family of connected graphs is just $\mathcal{C}_2$.

**Theorem 11.** For any $k \in \mathbb{N}$, the family $\mathcal{C}_{k+1}$ is complete.

**Proof.** Let $k \in \mathbb{N}$. It is trivial that $K_1 \in \mathcal{C}_{k+1}$, and it is routine to show that $\mathcal{C}_{k+1}$ is closed under contraction. Thus, to prove that $\mathcal{C}_{k+1}$ is complete, it remains to prove axiom (C3) for $\mathcal{C}_{k+1}$.

Let $H \in \mathcal{C}_{k+1}$. Suppose $G$ is a supergraph of $H$ such that $G/H \in \mathcal{C}_{k+1}$.

Now let $X_1, X_2, \ldots, X_k$ be $k$ even subsets of $V(G)$. Denote the vertex of $G/H$ corresponding to $H$ by $v_H$ and define, for $i \in \{1, 2, \ldots, k\}$,

$$X_i/H = \begin{cases} X_i - V(H) & \text{if } |V(H) \cap X_i| \text{ is even;} \\ X_i \cup \{v_H\} - V(H) & \text{if } |V(H) \cap X_i| \text{ is odd.} \end{cases} \quad (5)$$

Since $G/H \in \mathcal{C}_{k+1}$, there are $k$ disjoint sets $F_1, F_2, \ldots, F_k \subseteq E(G) - E(H)$ such that

$$O((G/H)[F_i]) = X_i/H \quad (1 \leq i \leq k) \quad (6)$$

Since $H \in \mathcal{C}_{k+1}$ there are disjoint sets $E_1, E_2, \ldots, E_k \subseteq E(H)$ such that

$$O(H[E_i]) = (O(G[F_i]) \cap V(H)) \Delta (V(H) \cap X_i), \quad (7)$$

where $\Delta$ denotes the symmetric difference. By (7),

$$O(G[E_i \cup F_i]) \cap V(H) = (O(G[E_i]) \Delta O(G[F_i])) \cap V(H)$$

$$= (O(H[E_i]) \Delta O(G[F_i])) \cap V(H) \quad (8)$$

$$= V(H) \cap X_i.$$

By $O(G[E_i]) \subseteq V(H)$, by (6), and by (5),
\[ O(G[E_i \cup F_i]) - V(H) = O(G[E_i]) \Delta O(G[F_i]) - V(H) \]
\[ = O(G[F_i]) - V(H) \]
\[ = O((G/H)[F_i]) - v_H \]
\[ = x_i / H - v_H = x_i - V(H). \]

By (8) and (9),
\[ O(G[E_i \cup F_i]) = x_i \quad (1 \leq i \leq k), \]
and since \(X_1, X_2, \ldots, X_k\) are arbitrary even subsets of \(V(G), \ G \in \mathcal{C}_{k+1}.\) Therefore, \(\mathcal{C}_{k+1}\) is complete. □

**Theorem 12.** For any \(p \in \mathbb{N}, \ \mathcal{C}_{2p+1} \subseteq \mathcal{Q}_{2p+1}.

**Proof.** Let \(p \in \mathbb{N}.\) Let \(H \in \mathcal{C}_{2p+1}\) and let \(G\) be a supergraph of \(H.\) It is routine to prove that if \(G \in \mathcal{Q}_{2p+1}\) then \(G/H \in \mathcal{Q}_{2p+1}.\) Thus, to prove \(H \in \mathcal{Q}_{2p+1},\)
we must assume that \(G/H \in \mathcal{Q}_{2p+1}\) and prove that \(G \in \mathcal{Q}_{2p+1}.

Suppose
\[ G/H \in \mathcal{Q}_{2p+1}. \]

Then we have a partition
\[ E(G/H) = F_1 \cup F_2 \cup \cdots \cup F_{2p+1} \]
such that
\[ O((G/H)[F_i]) = O(G/H) \quad (1 \leq i \leq 2p + 1). \]

Define
\[ W_i = O(G[F_i]); \quad (1 \leq i \leq 2p + 1), \quad (10) \]
and set
\[ V_i = W_i \Delta O(G) \quad (1 \leq i \leq 2p + 1). \quad (11) \]

Then
\[ V_i \subseteq V(H), \]

since any vertex of \(V(G) - V(H)\) is in \(W_i = O(G[F_i])\) if and only if it is in \(O((G/H)[F_i]) = O(G/H).\)
Since $H \in \mathcal{C}_{2p+1}$, there is a partition

$$E(H) = E_1 \cup E_2 \cup \cdots \cup E_{2p+1}$$

such that $E_i$ is a $V_i$-join in $H$ ($1 \leq i \leq 2p$). Hence by (10) and (11),

$$O(G[E_i \cup F_i]) = O(G[F_i]) \Delta O(G[F_i]) = V_i \Delta W_i = O(G),$$

for $1 \leq i \leq 2p$, and it follows that $O(G[E_{2p+1} \cup F_{2p+1}]) = O(G)$.

Hence, $G \in \mathcal{S}_{2p+1}$. Since $G$ is an arbitrary supergraph of $H$ such that $G/H \in \mathcal{S}_{2p+1}$, we have $H \in \mathcal{S}_{2p+1}^O$, and Theorem 12 is proved. \hfill \square

**Corollary 12A.** For all $p \in \mathbb{N}$,

$$\mathcal{S}_{2p+1}^0 \subseteq \mathcal{C}_{2p+1} \subseteq \mathcal{S}_{2p+1}^1 \subseteq \mathcal{S}_{2p+1}.$$ 

**Proof.** Let $G \in \mathcal{S}_{2p+1}^0$. Then $G$ has $2p$ edge-disjoint spanning trees, say $T_1, T_2, \ldots, T_{2p}$. Let $V_1, V_2, \ldots, V_{2p}$ be $2p$ even subsets of $V(G)$. Each tree $T_i$ contains a $V_i$-join ($1 \leq i \leq 2p$), and since these $V_i$-joins are edge-disjoint, $G \in \mathcal{C}_{2p+1}$. Hence $\mathcal{S}_{2p+1}^0 \subseteq \mathcal{C}_{2p+1}$, and by Theorems 12 and 4,

$$\mathcal{C}_{2p+1} \subseteq \mathcal{S}_{2p+1}^0 \subseteq \mathcal{S}_{2p+1}^1 \subseteq \mathcal{S}_{2p+1}.$$ \hfill \square

Jaeger [12] proved that $\mathcal{S}_4(4,0) \subset \mathcal{S}_2(2,0) \subset \mathcal{S}_2$, and in [4], [6], and [8] (combined), we improved that by proving (4). Corollary 12A is a similar improvement of an analogous result of Jaeger (Theorem 10).

**Theorem 13.** The kernel of $\mathcal{S}_3$ contains the four-cycle.

**Proof.** By Lemma 2, $\mathcal{S}_3$ is closed under contraction, and so it suffices to show that for a four-cycle $H$ and for any supergraph $G$ of $H$,

$$G/H \in \mathcal{S}_3 \Rightarrow G \in \mathcal{S}_3.$$ 

Let $H$ be the four-cycle $wxyzw$, and suppose

$$G/H \in \mathcal{S}_3.$$ 

Then there is a partition

$$E(G/H) = F_1 \cup F_2 \cup F_3,$$

such that $O((G/H)[F_i]) = O(G/H)(1 \leq i \leq 3)$. Hence,

$$G/H - F_i$$

is an even graph ($1 \leq i \leq 3$).
Define
\[ U_i = O(G - E(H) - F_i) \quad (1 \leq i \leq 3). \]

Hence, \( U_i \subseteq V(H) \). We have
\[ O(G[F_i]) = U_i \Delta O(G) \quad (1 \leq i \leq 3), \tag{12} \]
and since \( |O(G - E(H) - F_i)| \) is even,
\[ |U_i| \text{ is even} \quad (1 \leq i \leq 3). \tag{13} \]

**Case 1.** Suppose that \( H \) contains a \( U_1 \)-join \( E_1 \) and a \( U_2 \)-join \( E_2 \) such that \( E_1 \cap E_2 = \emptyset \). For \( i \in \{1, 2\} \), (12) gives
\[ O(G[E_i \cup F_i]) = O(G[E_i]) \Delta O(G[F_i]) = U_i \Delta U_i \Delta O(G) = O(G). \]
It follows that \( O(G[E(G) - (E_1 \cup F_1) - (E_2 \cup F_2)]) = O(G) \), and so \( G \in \mathcal{F}_3 \).

**Case 2.** Suppose Case 1 does not apply. Check that neither \( U_1 \) nor \( U_2 \) can be empty; check that no \( U_i \) \((i = 1, 2)\) consists of two adjacent vertices of \( H \); and check that if \( U_1 \) and \( U_2 \) are equal even sets, then Case 1 applies. Hence by (13) and by symmetry on the four-cycle \( H = wxyzw \), we are left with two possibilities:
(a) \( U_1 = V(H) \) and \( U_2 = \{w, y\}; \)
(b) \( U_1 = \{x, z\} \) and \( U_2 = \{w, y\}. \)

Since \( G - E(H) - F_3 = G[F_1 \cup F_2] \), (12) gives
\[ U_3 = O(G - E(H) - F_3) \]
\[ = O(G[F_1 \cup F_2]) = O(G[F_i]) \Delta O(G[F_i]) \]
\[ = U_1 \Delta U_2. \tag{14} \]
If (b) holds, then (14) would imply \( U_3 = V(H) \), and by a change of subscripts, this is equivalent to (a). Thus, without loss of generality, we assume that (a) holds and hence by (14) that
\[ U_3 = \{x, z\}. \]

By (12) and (a), \( O(G - F_i) = U_1 = V(H) \). Since each component of \( G - E(H) - F_i \) has evenly many odd-degree vertices, all in \( V(H) \), the component \( G_w \) (say) of \( G - E(H) - F_i \) containing \( w \) contains one or three members of \( \{x, y, z\}. \)

Subcase 2A. Suppose \( y \in V(G_w) \). Then there is a \((w, y)\)-path \( P \) (say) in \( G_w \subseteq G - E(H) - F_1 \). Thus, by (12),

\[
O(G[F_1 \Delta \{wx, yz\}]) = O(G[F_1]) \Delta \{w, x, y, z\} \\
= U_1 \Delta O(G) \Delta V(H) = O(G) ;
\]

\[
O(G[F_2 \Delta E(P)]) = O(G[F_2]) \Delta \{w, y\} \\
= U_2 \Delta O(G) \Delta \{w, y\} = O(G) ;
\]

\[
O(G[F_3 \Delta E(P) \Delta \{wz, xy\}]) = O(G[F_3]) \Delta \{w, y\} \Delta \{w, x, y, z\} \\
= U_3 \Delta O(G) \Delta \{x, z\} = O(G). 
\]

Also,

\[
(F_1 \Delta \{wx, yz\}) \cup (F_2 \Delta E(P)) \cup (F_3 \Delta E(P) \Delta \{wz, xy\})
\]

is a partition of \( E(G) \), and hence it is an \( O(G) \)-join of \( G \).

Subcase 2B. Suppose \( x \in V(G_w) \) (by symmetry, this is similar to the case \( z \in V(G_w) \)). Then there is a \((w, x)\)-path \( P \) (say) in \( G_w \subseteq G - E(H) - F_1 \). By (12), as in Subcase 2A,

\[
O(G[F_1 \Delta \{wx, yz\}]) = O(G) ;
\]

\[
O(G[F_2 \Delta E(P) \Delta \{xy\}]) = O(G[F_2]) \Delta \{w, y\} = O(G) ;
\]

\[
O(G[F_3 \Delta E(P) \Delta \{wz\}]) = O(G[F_3]) \Delta \{x, z\} = O(G). 
\]

As before, we have an \( O(G) \)-join in \( G \), formed by the partition

\[
(F_1 \Delta \{wx, yz\}) \cup (F_2 \Delta E(P) \Delta \{xy\}) \cup (F_3 \Delta E(P) \Delta \{wz\}) .
\]

In both subcases, we have \( G \in \mathcal{F}_3 \), as desired.
Therefore, \( H \in \mathcal{F}_3^O \). ●

Corollary 13A. Both \( \mathcal{CL} \subseteq \mathcal{F}_3 \) and

\[
\mathcal{C}_3 \cup \{C_4\} \subseteq \mathcal{F}_3^O \subseteq \mathcal{F}_3 .
\]

Proof. Suppose that \( G \in \mathcal{CL} \) and let \( V_1 \) and \( V_2 \) be even subsets of \( V(G) \).
Since \( G \in \mathcal{CL} \), there is a \( V_1 \)-join \( E_1 \) in \( G \) such that \( G - E_1 \) is connected. But \( G - E_1 \) thus has a \( V_2 \)-join, and so \( G \in \mathcal{C}_3 \). Therefore, \( \mathcal{CL} \subseteq \mathcal{C}_3 \). The latter part of Corollary 13A comes from Theorems 12 and 13. ●

It can be shown that each containment in Corollary 13A is strict. For example, \( Q_3 - v \) (the cube minus a vertex) is in \( \mathcal{C}_3 \) but not in \( \mathcal{CL} \).
6. THE MAIN RESULT

Theorem 14. Let $G$ be a graph. If $G$ is at most 5 edges short of being 4-edge-connected, then exactly one of the following holds:

(a) $G \in \mathcal{F}_3$;
(b) $G$ has at least one cut-edge;
(c) The $(\mathcal{EL} \cup \{C_4\})$-reduction of $G$ is the Petersen graph ($G$ is contractible to the Petersen graph).

Proof. It is easy to check that the conclusions are mutually exclusive, since the Petersen graph is not in $\mathcal{F}_3$.

By Corollary 13A, $\mathcal{EL} \cup \{C_4\} \subset \mathcal{F}_3^0$. Hence, we can define

$$\mathcal{T} = \mathcal{EL} \cup \{C_4\},$$

knowing that $\mathcal{T} \subseteq \mathcal{F}_3^0$ in Theorem 2. By Lemma 1, $\mathcal{EL}$ is complete, and so by Theorem 6, $\mathcal{EL}$ is closed under homomorphisms. Since every elementary homomorphism of $C_4$ is in $\mathcal{EL}$, $\mathcal{T}$ is thus closed under homomorphisms. By Lemma 2, $\mathcal{F}_3$ is closed under contractions, and so $\mathcal{T}$ and $\mathcal{F}_3$ satisfy the hypothesis of Theorem 2. Hence, the $\mathcal{T}$-reduction of any graph $G$ is unique, and by (2),

$$G \in \mathcal{F}_3 \iff G/\mathcal{T} \in \mathcal{F}_3.$$

By way of contradiction, suppose that Theorem 14 is false, and let $G$ be a counterexample with the fewest edges possible. Thus, we suppose that

$$G \in \mathcal{F}(4, 5)$$

and that (a), (b), and (c) of Theorem 14 are false. By Lemma 2, $\mathcal{F}(4, 5)$ and $\mathcal{F}_3$ are closed under contraction, and so if $G$ is a counterexample, then so is $G/\mathcal{T}$. By the minimality of $G$, therefore,

$$G \text{ is } \mathcal{T}\text{-reduced}. \tag{15}$$

This means that

$$G \text{ has no subgraph in } \mathcal{EL} \cup \{C_4\}. \tag{16}$$

Also, (a), (b), and (c) of Theorem 14 are false, i.e.,

$$G \notin \mathcal{F}_3 \text{ and } G \notin \{\text{Petersen graph}\} \tag{17}$$

and $\kappa'(G) \neq 1$. If $\kappa'(G) \geq 4$ then $G \in \mathcal{F}(4, 0) \subseteq \mathcal{F}_{2, 0} \subseteq \mathcal{EL}$, by (4). But since (16) says $G \in \mathcal{EL}^r$, Theorem 5 gives $G = K_1$, and so $G \in \mathcal{F}_3$, contrary
to (17). If $G$ is disconnected, then each component is in $\mathcal{S}(4, 5)$ and thus satisfies a conclusion of Theorem 14, by the minimality of $G$. But then $G$ satisfies Theorem 14, too. Hence,

$$\kappa'(G) \in \{2, 3\}. \quad (18)$$

**Lemma 3.** Let $G$ be a graph, let $v \in V(G)$, and let $e'$ and $e''$ be distinct edges incident with $v$. If $G(e', e'') \in \mathcal{S}_3$, then $G \in \mathcal{S}_3$. \hfill \blacksquare

This lemma is routine and its proof is omitted. (Recall that $G(e', e'')$ is the graph obtained from $G$ when $e'$ and $e''$ are lifted.)

**Lemma 4.** If $G$ is a $\mathcal{T}$-reduced graph with $\delta(G) \geq 3$, then the order of $G$ is at least $1 + 3\Delta(G)$. If also $\Delta(G) = 3$ and if $G$ has order at most 10, then $G$ is the Petersen graph.

**Proof.** Let $G$ be a graph with $\delta(G) \geq 3$ and let $v$ be a vertex with $d(v) = \Delta(G)$. Let $X_t = X_t(v)$ denote the set of vertices of $G$ at distance $t$ from $v$, where $t \geq 0$. Since $G$ is $\mathcal{T}$-reduced, $G$ has girth at least 5, by (16). Hence, each vertex of $X_t$ is connected to $v$ by a unique path of length $t$, when $t \in \{1, 2\}$. Therefore, $|X_1| = d(v) = \Delta(G)$, and $\delta(G) \geq 3$ implies $|X_2| \geq 2|X_1|$. Hence,

$$|V(G)| \geq |X_0| + |X_1| + |X_2| \geq 1 + 3|X_1| = 1 + 3\Delta(G).$$

Now suppose that $\Delta(G) = 3$ and that $G$ has order at most 10. By the inequality above, $G$ has order exactly 10, and since $G$ has no cycle of length less than 5, it is routine to show that equality can only hold when $G[X_2]$ is a 6-cycle such that $G$ is the Petersen graph. \hfill \blacksquare

**Proof of Theorem 14, continued.** By the hypothesis $G \in \mathcal{S}(4, 5)$, we have

$$|E(G)| \geq 2n - 5, \quad (19)$$

where $n$ is the order of $G$.

Since $G \neq K_2$ and by (19) and Theorem 9,

$$4n - 10 \leq \sum_{v \in V(G)} d(v) \leq 4n - 8. \quad (20)$$

For $k \in \mathbb{N} \cup \{0\}$, define

$$V_k = \{v \in V(G) | d(v) = k\}.$$
We claim that

\[ V_0 \cup V_1 \cup V_2 = \emptyset. \tag{21} \]

By (18), \( V_0 \cup V_1 = \emptyset \). Suppose \( v \in V_2 \) and let \( e \in E(G) \) be incident with \( v \). By Lemma 2, \( G \in \mathscr{S}(4,5) \) implies \( G/e \in \mathscr{S}(4,5) \). Thus, since \( G \) is a smallest counterexample, either \( G/e \in \mathscr{S}_3 \) (in which case \( G \in \mathscr{S}_3 \), a contradiction); or \( \kappa'(G/e) = 1 \) (whence \( \kappa'(G) = 1 \), a contradiction); or \( G/e \) is the Petersen graph (in which case \( G \notin \mathscr{S}(4,5) \), a contradiction). This proves (21).

Since (21) implies \( \delta(G) \geq 3 \) we have

\[ \sum_{v \in V(G)} d(v) = 4n - 8 \Rightarrow |V_3| \geq 8 \tag{22} \]

and

\[ \sum_{v \in V(G)} d(v) = 4n - 10 \Rightarrow |V_3| \geq 10. \tag{23} \]

By \( G \in \mathscr{S}(4,5) \),

\[ |V_3| \leq 10. \tag{24} \]

Since \( G \in \mathscr{S}(4,5) \) by hypothesis, it is possible to add a set \( E_5 \) (say) of 5 edges to \( G \) such that \( \kappa'(G + E_5) \geq 4 \).

Let \( \partial E_5 \) denote the set of vertices of \( V(G) \) incident with an edge of \( E_5 \). Since \( \Sigma d(v) \) is even, (20) implies that \( \Sigma d(v) \in \{4n - 8, 4n - 10\} \), and so either (22) or (23) applies. Hence, by (22), (23), and (24),

\[ 8 \leq |V_3| \leq 10, \]

and since \( \kappa'(G + E_5) \geq 4 \), we must have \( V_3 \subseteq \partial E_5 \) and hence

\[ |\partial E_3 \cap (V(G) - V_3)| = |\partial E_3| - |V_3| \leq 10 - 8 = 2. \tag{25} \]

We now give a proof (several pages long) that

\[ V_4 - \partial E_5 = \emptyset. \tag{26} \]

By way of contradiction, suppose \( G \) has a vertex \( v \in V_4 - \partial E_5 \). Since \( G \) is a minimum counterexample to Theorem 14, \( v \) is not a cutvertex. Let \( \{e_1, e_2, e_3, e_4\} \) denote the set of four edges of \( E(G) \) incident with \( v \). It follows from (16) that \( G \) is simple, and so we can apply Theorem 8 at \( v \). Since \( \kappa'(G + E_5) \geq 4 \), Mader's Lifting Theorem (Theorem 8) asserts that \( E(G) \) has two edges \( (e_1 \text{ and } e_3, \text{ say}) \)}
incident with \( v \) that can be lifted such that if \( G' \) denotes \( G(e_1, e_3) \) then

\[
\kappa'_{G+E_5}(x, y) = \kappa_{G+E_5}(x, y) \geq 4 \quad (\forall x, y \in V(G) - v).
\]

(27)

In \( G' \), \( d(v) = 2 \) and \( v \) is incident with \( e_2 \) and \( e_4 \). Let \( e_{13} \) denote the edge of \( G' \) created when \( e_1, e_3 \in E(G) \) are lifted. Denote by \( G_0 \) the graph obtained from \( G' \) by dissolving \( v \), and let \( e_{24} \) denote the edge of \( E(G_0) \) thus created to replace \( e_2 \) and \( e_4 \) and \( v \). Thus, \( e_{13}, e_{24} \in E(G_0) \).

By (27) and the definition of \( G_0 \), we have

\[
\kappa'(G_0 + E_5) \geq 4.
\]

**Case A.** Suppose \( \kappa'(G_0) \geq 2 \), i.e., \( G_0 \in \mathcal{S}(2, 0) \). Since \( G_0 \in \mathcal{S}(4, 5) \) and since \( G \) is a smallest counterexample in \( \mathcal{S}(4, 5) \cap \mathcal{S}(2, 0) \), either \( G_0 \in \mathcal{S}_3 \) or the \( \mathcal{T} \)-reduction of \( G_0 \) is the Petersen graph.

If \( G_0 \in \mathcal{S}_3 \) then \( G' \in \mathcal{S}_3 \), and by Lemma 3 (with \( e' = e_1 \) and \( e'' = e_3 \)), \( G \in \mathcal{S}_3 \), contrary to (17).

Hence, \( G_0/\mathcal{T} \) is the Petersen graph. Since \( G_0 + E_5 \in \mathcal{S}(4, 0) \) and since \( \mathcal{S}(4, 0) \) is closed under contraction (by Lemma 2),

\[
(G_0 + E_5)/\mathcal{T} \in \mathcal{S}(4, 0),
\]

and so each vertex of the Petersen graph \( G_0/\mathcal{T} \) is incident with exactly one edge of \( E_5 \). Let \( H_1, H_2, \ldots, H_{10} \) denote the ten maximal subgraphs of \( G_0 \) that lie in \( \mathcal{T} \) and that are each contracted to obtain from \( G_0 \) the ten vertices of \( G_0/\mathcal{T} \). We have

\[
|V(H_i) \cap \partial E_5| = 1 \quad (1 \leq i \leq 10),
\]

(28)

and each \( H_i \) is incident with exactly three edges of \( G_0/\mathcal{T} \) that have exactly one end in \( V(H_i) \). Each \( H_i \) not containing \( e_{13} \) or \( e_{24} \) is a subgraph of \( G \), and thus is \( \mathcal{T} \)-reduced, by (15) and since all subgraphs of a reduced graph are reduced. By Theorem 5, \( \mathcal{T} \cap \mathcal{T}^r = \{K_1\} \), and so such a subgraph \( H_i \) (not containing \( e_{13} \) or \( e_{24} \)) is \( K_1 \).

Let \( E = E(G_0/\mathcal{T}) \). Then \( E \subseteq E(G_0) \).

**Subcase A1.** Suppose \( e_{13}, e_{24} \in E \). Then each \( H_i \) (\( 1 \leq i \leq 10 \)) is a \( K_1 \), by our prior remarks, and so \( G_0 \) is the Petersen graph. By the construction of \( G_0 \) from \( G \), the graph \( G \) has order 11, and \( \delta(G) = 3 \), and \( G \) has a single vertex \( v \) of degree \( \Delta(G) = 4 \). But this violates Lemma 4, since \( G \) is \( \mathcal{T} \)-reduced.

**Subcase A2.** Suppose that some \( H_i \) (\( 1 \leq i \leq 10 \)) contains exactly one member of \( \{e_{13}, e_{24}\} \), say \( E(H_i) \cap \{e_{13}, e_{24}\} = \{e_{13}\} \). Since the Petersen graph \( G_0/\mathcal{T} \) is 3-regular, there are exactly three edges of \( G_0 \) with just one end in \( V(H_i) \). By (28), \( |V(H_i) \cap \partial E_5| = 1 \), and so there are exactly four edges of \( G_0 + E_5 \) with exactly one end in \( V(H_i) \). By \( \kappa'(G_0 + E_5) \geq 4 \),

\[
\kappa'(G_0 + E_5) \geq 4.
\]
\[
\sum_{v \in V(H)} d_{G_0 + E_3}(v) \geq 4|V(H)|
\]

and so

\[
\sum_{v \in V(H')} d_{H'}(v) \geq 4|V(H')| - 4.
\]

Hence, \(|E(H_i)| \geq 2|V(H)| - 2\). Let \(H'_i\) denote the graph obtained from \(H_i\) by replacing \(e_{13}\) with \(\{e_1, e_3, v_{13}\}\) so that \(H'_i\) is the corresponding subgraph of \(G'\) and of \(G\) where \(v = v_{13}\). Then

\[
|E(H'_i)| \geq 2|V(H'_i)| - 3.
\]

But this violates Theorem 9, for since \(G\) is \(\mathcal{F}\)-reduced, so is its subgraph \(H'_i\).

**Subcase A3.** Suppose that some \(H_i (1 \leq i \leq 10)\), say \(H_i = H\), contains both \(e_{13}\) and \(e_{24}\). Since the Petersen graph \(G_0/\mathcal{F}\) is 3-regular, there are exactly three edges of \(G_0\) with exactly one end in \(E(H)\). By (28), \(|V(H) \cap \partial E_3| = 1\), and so there are exactly four edges of \(G_0 + E_3\) with exactly one end in \(V(H)\). Hence, \(\kappa'(G_0 + E_3) \geq 4\) gives

\[
\sum_{v \in V(H)} d_{G_0 + E_3}(v) \geq 4|V(H)|
\]

and so

\[
\sum_{v \in V(H)} d_{H}(v) \geq 4|V(H)| - 4.
\]

Thus, \(|E(H)| \geq 2|V(H)| - 2\). Let \(H''\) be the graph obtained from \(H\) by replacing \(\{e_{13}, e_{24}\}\) with \(\{e_1, e_3, e_4, v\}\), so that \(H''\) is the subgraph of \(G\) corresponding to \(H\). Then \(|E(H'')| \geq 2|V(H'')| - 2\). This violates Theorem 9, for \(H''\) is \(\mathcal{F}\)-reduced, since it is a subgraph of the \(\mathcal{F}\)-reduced graph \(G\).

These subcases exhaust the possibilities and always yield contradictions, and so Case A is complete.

**Case B.** Suppose \(\kappa'(G_0) < 2\). By (18), \(\kappa'(G) \in \{2, 3\}\).

First we dispose of the case \(\kappa'(G) = 2\). Let \(\{e, e'\}\) be a 2-edge-cutset of \(G\), and let \(G_1\) and \(G_2\) denote the components of \(G - \{e, e'\}\). Define \(n_i = |V(G_i)|\), for \(i = 1, 2\). By (21), \(\delta(G) \geq 3\), and so \(G_i \notin \{K_1, K_2\}\). Hence by Theorem 9,

\[
|E(G_i)| \leq 2n_i - 4 \quad (i = 1, 2).
\]
By this and (19),
\[2n - 5 \leq |E(G)| = |E(G_1)| + |E(G_2)| + |\{e, e'\}| \leq (2n_1 - 4) + (2n_2 - 4) + 2 = 2n - 6,\]
a contradiction. Therefore, we may assume \(\kappa'(G) = 3\).

Since \(\kappa'(G) = 3\) and \(\kappa'(G_0) < 2\), the derivation of \(G_0\) from \(G\) implies that \(G_0\) has a cut-edge \(e\) (that is not a cut-edge of \(G\)), and that one component of \(G_0 - e - \{e_{13}, e_{24}\}\) (say \(G_1\)) contains both ends of \(e_{13}\), while the other component of \(G_0 - e - \{e_{13}, e_{24}\}\) (say \(G_2\)) contains both ends of \(e_{24}\). (See Figure 1.) For \(i \in \{1, 2\}\), let \(G_i(v)\) denote \(G[V(G_i) \cup \{v\}]\). In \(G_i(v)\), \(v\) is incident with \(e_i\) and \(e_{i+2}\) only. By Lemma 2, \(\mathcal{F}(4, 5)\) is closed under contraction, and so
\[G/G_i(v) \in \mathcal{F}(4, 5) \quad (i = 1, 2)\]
and by the minimality of \(G\), both \(G/G_1(v)\) and \(G/G_2(v)\) satisfy a conclusion of Theorem 14.

By (18) and since \(\mathcal{F}(2, 0)\) is closed under contraction (Lemma 2), neither \(G/G_1(v)\) nor \(G/G_2(v)\) has a cut-edge. We claim that neither has the Petersen graph as a \(\mathcal{T}\)-reduction, either.

Let \(i \in \{1, 2\}\). By way of contradiction, suppose that the \(\mathcal{T}\)-reduction of \(G/G_i(v)\) is the Petersen graph, and let \(E_{15}\) be the set of 15 edges of this Petersen graph. Then \(E_{15} \subseteq E(G)\) and \(G - E_{15}\) consists of 10 components, say \(H_1, H_2, \ldots, H_{10}\), one of which contains \(G_i(v)\). By (15) and since subgraphs of reduced graphs are reduced, Theorem 9 implies
\[|E(H_j)| \leq 2|V(H_j)| - 2 \quad (1 \leq j \leq 10), \quad (29)\]

![FIGURE 1. Three graphs of Case B.](image-url)
with equality if and only if \( H_j = K_1 \). Hence, by \((19)\) and \((29)\),

\[
2n - 5 \leq |E(G)| = |E_{15}| + \sum_{j=1}^{10} |E(H_j)|
\]

\[
\leq 15 + \sum_{j=1}^{10} (2|V(H_j)| - 2) = 2n - 5,
\]

and so equality holds in \((29)\) for \(1 \leq j \leq 10\). But then the component \( H_j \) containing the nontrivial subgraph \( G_i(v) \) is trivial, a contradiction.

Therefore, \( G/G_1(v) \) and \( G/G_2(v) \) must satisfy the first conclusion of Theorem 14: both are in \( \mathcal{F}_3 \) and both have 3-edge-colorings such that the union of any two color classes is an even graph. For \( i \in \{1, 2\} \), the vertex \( v \) has degree 3 in \( G/G_i(v) \), and so the three edges incident with \( v \) (namely, \( e_1, e_3, e \) and \( e_2, e_4, e \) in \( G/G_2(v) \) and \( G/G_1(v) \), respectively) have different colors. These two edge-colorings can be joined (so that \( e \) has the same color in both) to give a 3-edge-coloring of \( E(G) \) that proves \( G \in \mathcal{F}_3 \). This contradicts \((17)\). Case B is complete and \((26)\) is proved.

If \( G \) is 3-regular, then \((24)\) and Lemma 4 imply that \( G \) is the Petersen graph and thus violates \((17)\). Hence, \( \Delta(G) \geq 4 \). By \((21)\), \( \delta(G) \geq 3 \). Set

\[
W = \bigcup_{i=5}^{\infty} V_i.
\]

By \((20)\), \((21)\), and \((24)\), \( |W| \leq 2 \). By \((25)\) and \((26)\), \( |V_4| \leq 2 \). Since \( |V_3| \leq 10 \),

\[
|V(G)| = |V_3 \cup V_4| \leq 12, \text{ or } W \neq \emptyset \text{ and }
\]

\[
|V(G)| = |V_3 \cup V_4 \cup W| \leq 14,
\]

and either case contradicts Lemma 4, for \( G \) is \( \mathcal{T} \)-reduced with \( \delta(G) \geq 3 \) by \((15)\) and \((21)\).

Since every case leads to a contradiction, there is no smallest counterexample \( G \), and Theorem 14 is proved. ■

7. SOME CONJECTURES

Conjecture 2. If \( G \in \mathcal{F}_{2,2} \) then exactly one of the following holds:

(a) \( G \in \mathcal{C}_L \).
(b) The \( \mathcal{C}_L \)-reduction of \( G \) is either \( 2K_1 \) or \( K_2 \) or \( K_{2,t} \) for some \( t \in \mathbb{N} \).

Conjecture 3. If \( G \in \mathcal{F}_{2,2} \) then exactly one of the following holds:

(a) \( G \in \mathcal{F}_3^0 \).
(b) The \( \mathcal{F}_3^0 \)-reduction of \( G \) is \( 2K_1 \) or \( K_2 \) or \( K_{1,2} \).
Conjecture 3 would follow from Conjecture 2. Suppose \( G \in \mathcal{S}_{2,2} \) and suppose Conjecture 2 is true. If \( G \in \mathcal{CL} \), then (a) of Conjecture 3 holds, by Corollary 13A. Since \( \mathcal{CL} \subseteq \mathcal{S}_3^0 \) (Corollary 13A), the \( \mathcal{S}_3^0 \)-reduction of \( G \) is obtained from a \( \mathcal{CL} \)-reduction of \( G \), by contracting subgraphs in \( \mathcal{S}_3^0 \). If \( G/\mathcal{CL} \in \{2K_1, K_{2,1}, K_2\} \), then \( G/\mathcal{CL} \) is \( \mathcal{S}_3^0 \)-reduced, but if \( G/\mathcal{CL} = K_t \) (\( t \geq 2 \)), then from Theorem 13 we get \( G/\mathcal{CL} \in \mathcal{S}_3^0 \), whence \( G \in \mathcal{S}_3^0 \).

**Conjecture 4.** If \( G \in \mathcal{S}_{2,3} \), then exactly one of the following holds:

(a) \( G \in \mathcal{S}_3^0 \).

(b) \( G \) has at least one cut-edge.

(c) \( G \) is contractible to the Petersen graph.

It follows from Theorem 7 that \( \mathcal{S}(4,5) \subseteq \mathcal{S}_{2,3} \), and so the hypothesis of Conjecture 4 is weaker than the hypothesis of Theorem 14. By Theorem 4, \( \mathcal{S}_3^0 \subseteq \mathcal{S}_3 \), and since this containment is proper (large cycles are not in \( \mathcal{S}_3^0 \)), (a) of the conclusion of Conjecture 4 is stronger than (a) of Theorem 14.

The Blanuša snark [2] is a 3-regular graph of order 18 and girth 5 that does not satisfy any conclusion of Theorem 14. It is 9 edges short of being 4-edge-connected. We know of no graph at most 8 edges short of being 4-edge-connected that does not satisfy a conclusion of Theorem 14.

In [6], we conjectured that \( \mathcal{CL} = \mathcal{S}_3^0 \). By Theorem 13, the analogous conjecture \( \mathcal{C}_3 = \mathcal{S}_3^0 \) is false, because \( \mathcal{C}_3 \) does not contain the four-cycle.

**References**


