Graph Homomorphisms into the Five-Cycle

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We consider those edge-minimal graphs having no homomorphism into the five-cycle. We characterize constructively such graphs having the additional property that they contain no topological $K_4$ as a subgraph. © 1988 Academic Press, Inc.

1. INTRODUCTION

For simple graphs $G$ and $H$, we consider the graph homomorphism

$$\theta : G \to H,$$

(1)

where $\theta$ maps $V(G)$ into $V(H)$ and where $xy \in E(G)$ implies $\theta(x) \theta(y) \in E(H)$. When $H$ is a complete graph, the homomorphism $\theta$ is the usual coloring, and the chromatic number and achromatic number are special cases. (These numbers and homomorphisms are related by the Homomorphism Interpolation Theorem [7]. For a bound, see [3].)

When the homomorphism (1) exists, we shall call $\theta$ an $H$-coloring of $G$. If $G$ has an $H$-coloring, then we call $G$ $H$-colorable. If $G$ has no $H$-coloring, but for all $e \in E(G)$, $G - e$ has an $H$-coloring, we say that $G$ is $H$-critical. For example, a graph is $K_{n-1}$-critical in this sense if and only if it is chromatically $n$-critical in the usual sense (of [2], for example).

A graph $F$ is uniquely $H$-colorable if for any $H$-colorings $\theta_1$ and $\theta_2$ of $F$ there is an automorphism $\varnothing$ of $H$ such that $\varnothing \theta_1 = \theta_2$.

PROPOSITION 1. If $G$ is $H$-critical, then $G$ cannot be separated by a uniquely $H$-colorable subgraph $F$.

The proof is an imitation of the proof for the case $H = K_n$, i.e., for chromatically critical graphs. We omit the details.

An $(x, y)$-arc $A(x, y)$ of $G$ is a maximal path in $G$ whose ends are $x, \ y \in V(G)$ and whose interval vertices are divalent in $G$. Either $x$ and $y$...
are not divalent, or \( x = y \) and the component of \( G \) containing \( x \) is a cycle. An \((x, y)\)-arc \( A(x, y) \) having \( n \) edges will be denoted \( A_n(x, y) \). If \( A_n(x, y) \) is an arc of \( G \), then \( G - A_n(x, y) \) will denote the subgraph of \( G \) obtained by removing all edges and internal vertices of \( A_n(x, y) \).

**Proposition 2.** If \( G \) is \( C_{2k+1} \)-critical, then no arc of \( G \) has more than \( 2k - 1 \) edges.

Since the proof is routine, we omit it. We shall refer to both propositions in the next section.

We define an odd-\( TK_4 \) to be a \( TK_4 \) which, when embedded in the plane, has all four faces of odd girth. An odd-\( K_3^3 \) is defined to be any graph consisting of three edge-disjoint odd cycles \( C, C', C'' \), and three arcs

\[
\begin{align*}
A(u, u') & \quad (u \in V(C), \quad u' \in V(C')), \\
A(v', v'') & \quad (v' \in V(C'), \quad v'' \in V(C'')), \\
A(w'', w) & \quad (w'' \in V(C''), \quad w \in V(C)),
\end{align*}
\]

whose internal vertices have degree 2. (The graph \( R \) of Fig. 1 is an example of an odd-\( K_3^3 \) in which all three arcs have length 0.)

Dirac [5] proved that if a graph has no \( C_3 \)-coloring, then it has a \( TK_4 \). We [4] showed that the \( TK_4 \) in the conclusion of Dirac's theorem could be chosen to be an odd-\( TK_4 \). Gerards [6], in strengthening a result of [1], proved the following result:

**Theorem 1.** Let \( G \) be a graph with odd girth \( 2k + 1 \). Either \( G \) has a \( C_{2k+1} \)-coloring, or \( G \) contains an odd-\( TK_4 \) or an odd-\( K_3^3 \).

In this paper, we shall characterize constructively the graphs with no \( C_3 \)-coloring and no \( TK_4 \) subgraph.

### 2. The Main Results

The branch graph \( B(G) \) of a graph \( G \) (\( G \) not a cycle) is the multigraph obtained from \( G \) by replacing every arc by an edge joining its ends. A graph is nodally 3-connected if its branch graph is 3-connected (this is equivalent to Tutte's definition [8]). For an induced subgraph \( H \) of \( G \), the vertices of attachment of \( H \) in \( G \) are those vertices of \( H \) incident with at least one edge of \( E(G) - E(H) \).

We use \( d(u, v) \) to denote the distance in \( C_5 \) between \( u, v \in V(C_5) \). For \( x, y \in V(H) \), define

\[
D(x, y, H) = \{ d(\theta(x), \theta(y)) \mid \text{\( \theta \) is a \( C_5 \)-coloring of \( H \)} \}.
\]
Of course, \( \theta \) runs over all \( C_5 \)-colorings of \( H \). Thus,

\[
D(x, y, H) \subseteq \{0, 1, 2\}.
\]

Given two copies \( C, C' \) of \( C_5 \), with distinguished vertices \( x, z \in V(C) \) at distance 2 in \( C \), and with distinguished vertices \( y, z' \in V(C') \) at distance 2 in \( C' \), we denote by \( R_{xy} \) the nine-vertex graph obtained from \( C \cup C' \) by identifying \( z = z' \). See Fig. 1.

We shall denote by \( H + A_n(x, y) \) the graph obtained by adding to \( H \) an \((x, y)\)-arc \( A_n(x, y) \) having \( n \) edges, where \( x, y \in V(H) \). Denote (see Figs. 1 and 2)

\[
R'(x, y) = R_{xy} + A_2(x, y),
\]

\[
R''(x, y) = R_{xy} + A_3(x, y),
\]

and

\[
R_0(x, v) = R_{xy} + A_3(y, y), \quad v \in V(A_5(y, y)), \quad d(v, y) = 2.
\]

Thus, \( R_0(x, v) \) consists of three blocks, each a 5-cycle, and \( x, y, v \) are distinguished vertices, with \( y \) as a cutvertex.

An incremental subgraph \( H \) of a graph \( G \) is an induced subgraph \( H \) either isomorphic to \( R'(x, y) \) or \( R''(x, y) \) and with vertices of attachment \( \{x, y\} \) in \( G \), or isomorphic to \( R_0(x, v) \), with vertices of attachment \( \{x, v\} \) in \( G \), where \( v \in V(A_5(y, y)) \subseteq V(R_0(x, v)) \) is at distance 2 from \( y \).

**Theorem 2.** If \( G \) is a \( C_5 \)-critical graph with no \( TK_4 \) subgraph, and if \( G \) is neither \( K_3 \) nor \( R \), then \( G \) contains two edge-disjoint incremental subgraphs.
Proof. Throughout this proof, \( G \) will denote a \( C_5 \)-critical graph, neither \( K_3 \) nor \( R \), and without a \( TK_4 \).

Suppose that \( G \) is nodally 3-connected. Then the underlying branch graph \( B(G) \) is 3-connected. Therefore, there are vertices \( x, y \) of degree at least 3 in \( G \), and there are three internally disjoint \( (x, y) \)-paths \( P_1, P_2, P_3 \) in \( G \), by Menger's theorem. Also, \( B(G)-\{x, y\} \) is connected, and hence some path \( P_4 \) in \( G \) joins internal vertices of two of \( P_1, P_2, P_3 \). Then \( P_1 \cup P_2 \cup P_3 \cup P_4 \) is a \( TK_4 \) subgraph of \( G \).

Hence, by Proposition 1, we can assume that \( G \) has connectivity and nodal connectivity 2. We shall also suppose inductively, for the remainder of the proof, that for any \( C_5 \)-critical graph \( G' \notin \{K_3, R\} \), with \( |V(G')| < |V(G)| \), where \( G' \) has no \( TK_4 \), there are two edge-disjoint incremental subgraphs in \( G' \). As a basis for induction, note that if \( |V(G)| \leq 5 \), then the induction hypothesis holds vacuously.

We shall prove some lemmas next. In these lemmas, unions and intersections are defined as in [2].

**Lemma 1.** If \( G_{xy} \) is a 2-connected subgraph of \( G \) with vertices of attachment \( \{x, y\} \) in \( G \), where \( G_{xy} \neq K_2 \), then \( G_{xy} \) can be decomposed into connected subgraphs \( H, H' \) such that

\[
H \cup H' = G_{xy}, \quad H \cap H' = \{x, y\}.
\]

**Proof.** Since \( G_{xy} \) is 2-connected with vertices of attachment \( \{x, y\} \) in \( G \), there are internally disjoint \( (x, y) \)-paths \( P, P' \) in \( G_{xy} \). Since \( G \) is 2-connected, \( G - E(G_{xy}) \) has an \((x, y)\)-path \( P_0 \). If a path \( P'' \) in \( G_{xy} - \{x, y\} \) joins an internal vertex of \( P \) to an internal vertex of \( P' \), then \( P_0 \cup P \cup P' \cup P'' \) is a \( TK_4 \) in \( G \), contrary to the hypothesis of the theorem. Hence, no such path \( P'' \) exists, and so \( \{x, y\} \) separates \( G_{xy} \), and subgraphs \( H \) and \( H' \) exist as described, where \( P \subseteq H, P' \subseteq H' \).

**Lemma 2.** An acyclic subgraph of \( G \) with only two vertices of attachment \( \{u, v\} \) in \( G \) is a \((u, v)\)-path.

**Proof.** An acyclic subgraph \( H \) of \( G \) is a tree. Since \( G \) has no cutvertex (by Proposition 1), each vertex of degree 1 in \( H \) is a vertex of attachment in \( G \). Since \( H \) has only two vertices of attachment \( (u \text{ and } v) \) in \( G \), \( H \) must be a \((u, v)\)-path.

**Lemma 3.** There exist \( x, y \in V(G) \) and connected subgraphs \( H_1 \) and \( H_2 \) of \( G \), such that

\[
G = H_1 \cup H_2, \quad \{x, y\} = H_1 \cap H_2, \quad (2)
\]
and such that

\[ H_1 \text{ and } H_2 \text{ each contain at least one cycle.} \] (3)

Proof. Since the nodal connectivity of \( G \) is 2, and since \( G \) is not a cycle, the underlying branch graph \( B(G) \) has a separating set \( \{x, y\} \). Therefore, connected subgraphs \( H_1 \) and \( H_2 \), satisfying (2), exist, where \( H_1 \) and \( H_2 \) both have vertices of degree at least 3 and different from \( x \) and \( y \). If \( H_i \) has no cycle, then by Lemma 2, \( H_i \) is an \((x, y)\)-path, a contradiction. Therefore, \( H_1 \) and \( H_2 \) each contain a cycle. □

Since \( G \) is \( C_5 \)-critical, some graph \( H_i \ (i \in \{1, 2\}) \) of Lemma 3 satisfies \(|D(x, y, H_i)| = 1\), and so we lose no generality in assuming that

\[ |D(x, y, H_1)| = 1 \] (4)

and

\[ H_1 \text{ is maximal with respect to (2), (3), and (4).} \] (5)

Any ordered pair \((H_1, H_2)\) of induced subgraphs of \( G \) satisfying (5) (and hence (2), (3), and (4)) for some separating set \( \{x, y\} \) will be called a proper pair of subgraphs of \( G \).

Clearly, for \( i \in \{1, 2\} \), since \( G \) is 2-connected,

All cutvertices of \( H_i \) lie on a single \((x, y)\)-path. \hspace{1cm} (6)

Let \( H_0 \) be a 2-connected induced subgraph of \( G \) with vertices of attachment \( \{u, v\} \) in \( G \). If

\[ D(u, v, H_0) = \{0\}, \]

then \( H_0 \) is called a zero-block.

Lemma 4. If \((H_1, H_2)\) is a proper pair, then \( H_2 \) has no zero-block.

Proof. Suppose that \( H_{uv} \) is a zero-block of \( H_2 \). By the definition of a zero-block,

\[ D(u, v, H_{uv}) = \{0\}. \] (7)

By Lemma 1, \( H_{uv} \) has subgraphs \( H, H' \) such that

\[ H_{uv} = H \cup H', \quad \{u, v\} = H \cap H'. \]

Since \( G \) is \( C_5 \)-critical, (7) implies

\[ D(u, v, G - (H_{uv} - \{u, v\})) \subseteq \{1, 2\}, \]
and the values of $D(u, v, G - (H - \{u, v\}))$ and $D(u, v, G - (H' - \{u, v\}))$ are $\{1\}$ and $\{2\}$ in some order. Hence, $H$ or $H'$ could have been chosen in place of $H_2$ in (2) and (3), unless both $H$ and $H'$ are acyclic. This contradicts the maximality of $H_1$ in (4) and (5), except when both $H$ and $H'$ are acyclic. In the latter case, by Lemma 2, they are $(u, v)$-paths, and thus $H_{uv}$ is a cycle. But then (7) is false, a contradiction. 

**Lemma 5.** If $(H_1, H_2)$ is a proper pair, then $H_2$ has a cutvertex.

**Proof.** Suppose, by way of contradiction, that $H_2$ is 2-connected. By (3), $H_2 \neq K_2$. By Lemma 1, there are subgraphs $H, H'$ of $H_2$, such that

$$H_2 = H \cup H', \quad \{x, y\} = H \cap H'.$$

Since $G$ is $C_5$-critical and $D(x, y, H_1)$ is a singleton (by (4)), we have identical singletons

$$D(x, y, H_1 \cup H) = D(x, y, H_1 \cup H').$$

But this implies that $G$ has a $C_5$-coloring, a contradiction. Therefore, $H_2$ has at least one cutvertex. 

**Lemma 6.** If $(H_1, H_2)$ is a proper pair, then $H_2 = R_{xy}$, and $D(x, y, H_1) = \{2\}$.

**Proof.** Let $H_x$ (resp., $H_y$) be the block of $H_2$ containing $x$ (resp., $y$). By Lemma 5, $H_x \neq H_y$. Let $\{x, x'\}$ (resp., $\{y, y'\}$) be the vertices of attachment of $H_x$ (resp., of $H_y$) in $G$.

If

$$|D(x, x', H_x)| = |D(y, y', H_y)| = 2,$$

then $G$ has a $C_5$-coloring, a contradiction. Hence, there is no loss of generality in our supposing that

$$D(x, x', H_x) = \{t\}, \quad (8)$$

and Lemma 4 implies $t \in \{1, 2\}$. By (4), $|D(x, y, H_1)| = 1$.

**Case 1.** Suppose $D(x, y, H_1) = \{0\}$. Define $H'_1 = H_1 \cup H_x$, and note that (8) implies

$$D(x, y, H_1 \cup H_x) = \{t\}.$$

By the maximality of $H_1$ in (5), the induced subgraph $H'_2 = H_2 - (H_x - x')$ is acyclic with vertices of attachment $\{x', y\}$, and so by Lemma 2, $H'_2$ is an $(x', y)$-path $P$. Let $yz \in E(P)$ be the edge incident with $y$. Then
$D(x, z, H_1 \cup yz) = \{1\}$, and the graph induced by $H_1 \cup yz$ contradicts the maximality of $H_1$ in (5).

**Case 2.** Suppose $D(x, y, H_1)$ is $\{1\}$ or $\{2\}$. By (6), the blocks of $H_1$ and the blocks of $H_2$ are arranged in cyclic order in $G$. By Lemma 4 and the condition of Case 2, if $H_2$ has two or more cutvertices (and hence at least three blocks), then $G$ has a $C_5$-coloring. Hence, $H_2$ has a unique cutvertex $z = x' = y'$.

Since $G$ has no $C_5$-coloring, there is no triple $a_1, a_x, a_y \in \{1, 2\}$ with

$$a_1 \in D(x, y, H_1), \quad a_x \in D(x, z, H_x), \quad a_y \in D(y, z, H_y)$$

such that for some choice of plus and minus signs, chosen independently,

$$a_1 \pm a_x \pm a_y \equiv 0 \pmod{5}. \quad (9)$$

The absence of zero-blocks implies $0 \notin \{a_1, a_x, a_y\}$. If any of $D(x, y, H_1), D(x, z, H_x), D(y, z, H_y)$ has more than one member, then (9) has a solution, a contradiction. If all three sets have exactly one member, then since (9) has no solution, all are $\{1\}$ or all are $\{2\}$.

Suppose that for some $k \in \{1, 2\}$, we have $k = a_1, k = a_x, k = a_y$. Then

$$D(x, z, H_1 \cup H_y) = \{0, 3 - k\}. \quad (10)$$

By Lemma 1, the block $H_x$ can be decomposed into connected subgraphs $H, H'$, where

$$H_x = H \cup H', \quad \{x, z\} = H \cap H'.$$

Since $G$ is $C_5$-critical, it follows from (10) that $D(x, z, H_1 \cup H_y \cup H)$ and $D(x, z, H_1 \cup H_y \cup H')$ are $\{0\}$ and $\{3 - k\}$ in some order.

If $H'$ contains a cycle, then $H_1 \cup H_y \cup H$ would violate the maximality of $H_1$ in (5), since $H'$ could replace $H_2$ in (3). Therefore, $H'$ is acyclic. Likewise, $H$ is acyclic. By Lemma 2, both $H$ and $H'$ are $(x, z)$-paths. Since $G$ is $C_5$-critical, the lengths of $H$ and $H'$ are less than four, by Proposition 2, and they are unequal. By Proposition 1, $x$ and $z$ are not adjacent. Hence, one of $H, H'$ has length 2 and the other has length 3, and so $H_x$ is a 5-cycle, and $k = 2$.

A similar argument shows that $H_y$ is a 5-cycle, with $(x, z)$-arcs of lengths 2 and 3. Therefore, $H_2 = H_x \cup H_y = R_{xy}$, and $D(x, y, H_1)$ must be $\{2\}$. Lemma 6 is proved.

**Lemma 7.** If $(H_1, H_2)$ is a proper pair of subgraphs of $G$, and if $H_1$ is 2-connected, then either Theorem 2 holds for $G$ or there are subgraphs $H, H'$ of $H_1$ such that

$$H_1 = H \cup H', \quad \{x, y\} = H \cap H', \quad H = A_2(x, y),$$

where $\{x, y\}$ is the set of vertices of attachment of $H$, of $H'$, and $H_2$ in $G$. 
Proof. By Lemma 1, since \( H_1 \) is 2-connected, there are subgraphs \( H, H' \) such that
\[
H_1 = H \cup H', \quad \{x, y\} = H \cap H'.
\]
Since \( G \) is \( C_5 \)-critical, the three sets \( D(x, y, H_2), D(x, y, H), \) and \( D(x, y, H') \) are distinct subsets of \( \{0, 1, 2\} \) such that none of the three sets contains another one of the three sets. By Lemma 6,
\[
D(x, y, H_2) = \{0, 1\},
\]
and so we lose no generality in supposing
\[
D(x, y, H) = \{0, 2\}, \quad D(x, y, H') = \{1, 2\}.
\]
Therefore, \( H + A_1(x, y) \) is \( C_5 \)-critical and has no \( TK_4 \), and by the induction hypothesis, either \( H + A_1(x, y) = K_3 \), whence \( H = A_2(x, y) \), as required by Lemma 7, or \( H \) contains an incremental subgraph \( F \) of \( G \). It remains to exclude the latter case.

Suppose that \( H \) has an incremental subgraph \( F \). Let \( G' \) denote the graph obtained from \( G \) upon the replacement of \( H \) by \( A_2(x, y) \). Since \( G \) is \( C_5 \)-critical and \( D(x, y, H) = D(x, y, A_2(x, y)) \), the smaller graph \( G' \) is \( C_5 \)-critical. By the induction hypothesis, \( G' \) has two edge-disjoint incremental subgraphs, or \( G' = R \). In the former case, \( G' - (A_2(x, y) - x - y) \) has an incremental subgraph \( F' \), and so \( F \) and \( F' \) are two edge-disjoint incremental subgraphs of \( G \). In the latter case, since \( G' \) includes both \( H_2 = R_{xy} \) (by Lemma 6) and \( A_2(x, y) \), we must have \( H' = A_3(x, y) \). Hence, \( H' \cup H_2 = R''(x, y) \) is an incremental subraph \( F' \) of \( G \) that is edge-disjoint from \( F \). Thus, if \( H \) has an incremental subgraph \( F \), then the theorem holds. 

Lemma 8. If \( H_0 \) is a zero-block of \( G \), with
\[
|V(H_0)| \leq |V(G)| - 3,
\]
then \( H_0 \) contains an incremental subgraph.

Proof. Let \( u, v \in V(H_0) \) be the vertices of attachment of \( H_0 \). Since \( H_0 \) is a zero-block and \( G \) is \( C_5 \)-critical with no \( TK_4 \), \( H_0 + A_3(u, v) \) is also \( C_5 \)-critical with no \( TK_4 \). Since \( |V(H_0)| \leq |V(G)| - 3 \), the induction hypothesis applies to \( H_0 + A_3(u, v) \), and so \( H_0 \) contains an incremental subgraph.

Proof of Theorem 2 (continued). By (5), there is a proper pair \( (H_1, H_2) \) of incremental subgraphs of \( G \), and by Lemma 6,
\[
G = H_1 \cup H_2, \quad \{x, y\} = H_1 \cap H_2,
\]
and $H_2 = R_{xy}$ is a pair of 5-cycles with exactly one vertex $z$ in common, where $xz, yz \notin E(G)$.

**Case 1.** Suppose that $H_1$ is 2-connected. By Lemma 7, there are subgraphs $H, H'$ of $H_1$ such that

$$H_1 = H \cup H', \quad \{x, y\} = H \cap H', \quad H = A_2(x, y).$$

Let $t$ be the number of cutvertices of $H'$, and denote $x = z_0, y = z_{t+1}$. By (6), we can let $z_1, z_2, \ldots, z_t$ denote the $t$ cutvertices of $H'$ as they occur along an $(x, y)$-path in $H'$.

Since $H = A_2(x, y)$ and since $H_2 = R_{xy}$, the subgraph $F = H \cup H_2$ is an incremental subgraph $R'(x, y)$ in $G$.

We denote by $B_0, B_1, \ldots, B_t$ the $t+1$ blocks of $H'$, where

$$z_i, z_{i+1} \in V(B_i), \quad (0 \leq i \leq t).$$

If $H'$ is acyclic, then by Lemma 2, $H'$ is an $(x, y)$-path and since $G$ is $C_5$-critical and $D(x, y, H \cup H_2) = \{0\}$, we must have $H' = A_3(x, y)$ and hence $G = R$, contrary to our assumption. Therefore, $H'$ contains a cycle, and since $D(x, y, H \cup H_2) = \{0\}$, we have a proper pair $(H_3, H_4)$ satisfying

$$H \cup H_2 \subseteq H_3, \quad H_4 \subseteq H', \quad H_3 \cup H_4 = G, \quad H_3 \cap H_4 = \{z_j, z_k\},$$

for some $j$ and $k$ with $j < k$. By Lemma 6,

$$H_4 = B_j \cup B_{j+1} = R_{uv}, \quad \text{for} \quad u = z_j, v = z_{j+2} = z_k,$$

and $D(u, v, H_3) = \{2\}$. Therefore, $t \geq 2$, and $B_i$ is a 5-cycle for some $i$ such that $1 \leq i \leq t-1$, and so the proper pair $(H_3, H_4)$ may be chosen so that either

$$H \cup H_2 \cup B_0 \subseteq H_3 \quad \text{or} \quad H \cup H_2 \cup B_t \subseteq H_3,$$

without violating the requirement (3) that $H_4$ contain a cycle. If for some $h$ ($0 \leq h \leq t$), $B_h$ is a zero-block, then by Lemma 8 and the existence of $F$, $G$ has two edge-disjoint incremental subgraphs. Hence, we may assume that no $B_h$ is a zero-block. Consequently, $H_3 = H \cup H_2 \cup B_0$ and $H_3 = H \cup H_2 \cup B_t$ are two possible values of $H_3$ satisfying (5). By Lemma 6, $H_4 = R_{uv}$, where $\{u, v\}$ is $\{z_0, z_2\}$ or $\{z_{t-1}, z_{t+1}\}$, and $t = 2$, since $G$ is $C_5$-critical. Hence, $H' = R_0(x, y)$ and $F$ are two incremental subgraphs of $G$.

**Case 2.** Suppose that $H_1$ is not 2-connected. Thus, $H_1$ has at least one cutvertex $v \notin \{x, y\}$. By (6), all cutvertices of $H_1$ must lie on a single $(x, y)$-path in $H_1$. 
If $H_1$ has at least two zero-blocks, then by Lemma 8, $G$ has two incremental subgraphs. Hence, we can assume that $H_1$ has at most one zero-block. By (4), at most one block of $H_1$ is not a zero-block. It follows that $H_1$ has just a single cutvertex $v$, and so we shall denote by $H_{ux}$ and $H_{vy}$ the two blocks of $H_1$, where $v$ and $x$ are the two vertices of attachment of $H_{ux}$ in $G$, and $v$ and $y$ are the two vertices of attachment of $H_{vy}$ in $G$. Without loss of generality,

$$D(v, y, H_{vy}) = \{0\}, \quad (11)$$

and so by Lemma 6 and (11),

$$D(v, x, H_{ux}) = D(x, y, H_{ux} \cup H_{vy}) = D(x, y, H_1) = \{2\}.$$  

By Lemma 8 and (11), $H_{vy}$ has an incremental subgraph $F_1$.

Denote by $H_5$ the graph obtained from $H_{ux}$ by adding $R_{ux}$ and identifying both vertices named $v$ and identifying both named $x$. Note that $H_5$ is $C_5$-critical and $H_5$ has no $TK_4$ subgraph. Also, $H_5 \neq K_3$.

If $H_5 = R$, then $H_{ux} = C_5$, and so $F_2 = H_{ux} \cup H_2 = R_0(v, y)$ is an incremental subgraph of $G$. Then $F_1$ and $F_2$ are incremental subgraphs.

Suppose, instead, that $H_5 \neq R$. By the induction hypothesis, $H_5$ has two edge-disjoint incremental subgraphs, say $F_3$ and $F_4$. If either one, say $F_3$, is contained in $H_{ux}$, then $F_1$ and $F_3$ are two edge-disjoint incremental subgraphs of $G$. If neither $F_3$ nor $F_4$ is contained in $H_{ux}$, then $F_3$ and $F_4$ are $R_0$-type incremental subgraphs of $H_5$, but since $H_{ux}$ is one block, this is a contradiction.

Therefore, $G$ has two incremental subgraphs, and the induction is complete. Theorem 2 is proved.

**Theorem 3.** The graph $G$ is $C_5$-critical and has no $TK_4$ if and only if $G$ is obtained from $K_3$ by repeated applications of the following three operations:

1. The replacement of an arc $A_3(x, v)$ by $R_0(x, v)$ (where $x, v$ of the graph are identified with the corresponding distinguished vertices $x, v$ of $R_0(x, v)$).

2. The replacement of an edge $xy$ by $R''(x, y)$.

3. The replacement of vertex $u$ by nonadjacent vertices $x, y$, the joining of every neighbor of $u$ to exactly one of $x, y$, and the addition of $R'(x, y)$ such that no $TK_4$ subgraph is created.

In operations 2 and 3, the distinguished vertices $x, y$ of $R''(x, y)$ or $R'(x, y)$ are identified with the corresponding vertices with the same label in the graph.
EXAMPLE. The graph \( R \) can be obtained from \( K_3 \) by a single application of any one of these three operations. See Fig. 1 and 2.

Proof of Theorem 3. By Theorem 2, if \( G \) is \( C_5 \)-critical and has no \( TK_4 \) subgraph, then \( G \) has an incremental subgraph \( R_0(x, v) \), \( R'(x, y) \), or \( R''(x, y) \). By reversing one of the operations of Theorem 3 on this incremental subgraph, we obtain another \( TK_4 \)-free \( C_5 \)-critical graph \( G' \) with 9 fewer vertices and 12 fewer edges. By Theorem 2, \( G \) can be thus reduced to \( R \) and \( K_3 \), and so inductively we have

\[
|V(G)| \equiv 3 \quad (\mod 9) \tag{12}
\]

and

\[
3 |E(G)| + 3 = 4 |V(G)|. \tag{13}
\]

Conversely, let \( G' \) be a \( C_5 \)-critical graph with no \( TK_4 \) subgraph. Then \( G' \) satisfies (12) and (13). Moreover, one can prove inductively that \( D(x, y, G' - xy) = \{0, 2\} \) for all edges \( xy \) of \( G' \). Let \( G \) be a graph obtained from \( G' \) by one of the three operations of the theorem. Clearly, \( G \) has no \( TK_4 \), and so it remains to show that \( G \) is \( C_5 \)-critical.

Operation 1 replaces a subgraph \( A_3(x, v) \) satisfying

\[
D(x, v, A_3(x, v)) = \{1, 2\}
\]

with the subgraph \( R_0(x, v) \) having the property

\[
D(x, v, R_0(x, v)) = \{1, 2\},
\]

and since \( G' \) is \( C_5 \)-critical, so is \( G \). Operation 2 replaces the edge-subgraph \( xy \), satisfying

\[
D(x, y, xy) = \{1\},
\]

with the larger subgraph \( R''(x, y) \), such that

\[
D(x, y, R''(x, y)) = \{1\}.
\]

We claim that the graph \( G \) resulting from operation 2 is also \( C_5 \)-critical. By the above remark,

\[
D(x, y, G' - xy) = \{0, 2\},
\]

and for any proper spanning subgraph \( H \) of \( G' - xy \), \( 1 \in D(x, y, H) \), and so \( G[E(H) \cup E(R''(x, y))] \) has a \( C_5 \)-coloring. Also, if \( e \in E(R''(x, y)) \), then \( G - e \) has a \( C_5 \)-coloring. Thus, \( G \) is \( C_5 \)-critical, as claimed.
Let $G$ be obtained from a $C_5$-critical graph $G'$ by operation 3. Let

$$G_{xy} = G - (R'(x, y) - \{x, y\}).$$

In Operation 3 we replace a vertex $u \in V(G')$, where $D(u, u, u) = \{0\}$, by attaching $R'(x, y)$ to $G_{xy}$, where

$$D(x, y, R'(x, y)) = \{0\},$$

and so the resulting graph $G$ is not $C_5$-colorable. If $e \in E(G')$, then there is a $C_5$-coloring of $G' - e$, and it can be extended to a $C_5$-coloring of $G - e$. Hence, $E(G')$ is contained in a $C_5$-critical subgraph $H$ of $G$. We must have

$$|V(G')| < |V(H)| \leq |V(G)|, \tag{14}$$

and

$$|E(G')| < |E(H)| \leq |E(G)|, \tag{15}$$

and since (12) and (13) force equalities in (14) and (15), we have $H = G$. Thus, $G$ is $C_5$-critical, as claimed.

From (12) and (13), we get:

**Corollary.** If $G$ is $C_5$-critical and has no $TK_4$ subgraph, then

$$3|E(G)| + 3 = 4|V(G)|,$$

and

$$|V(G)| \equiv 3 \quad (\text{mod } 9).$$

**Theorem 4.** If $G$ is obtained from $R$ by repeated applications of the three operations of Theorem 3, then $G$ is $R$-colorable.

*Proof by Induction.* $R$ is $R$-colorable.

Suppose that $G'$ has an $R$-coloring $	heta'$, and that $G$ is obtained from $G'$ by a single application of one of the three operations of Theorem 3.

Define an $R$-coloring $	heta$ of $G$ by setting $\theta = \theta'$ on $G' - A_3(x, z)$ (operation 1), $G' - xy$ (operation 2), or $G' - u$ (operation 3), respectively, depending upon which operation is used to obtain $G$ from $G'$. It is easy to verify that $\theta$ can be extended to the incremental subgraph that is added to $G'$ to form $G$, such that $\theta$ becomes a homomorphism of $G$ onto $R$.

Next, we show that Theorem 2 is best-possible.

**Theorem 5.** There are infinitely many $C_5$-critical $TK_4$-free graphs with exactly two incremental subgraphs.
Proof. Let \( t \geq 1 \), and let \( H \) be the graph consisting of the edge-disjoint incremental subgraphs \( F_1 = R'(x_1, y_1) \) and \( F_2 = R_0(x, y) \), and, if \( t \geq 3 \), then \( 2t - 4 \) isolated vertices \( \{x_2, y_2, x_3, y_3, \ldots, x_{t-1}, y_{t-1}\} \). Thus, \( F_1 \cap F_2 = \{x_1, y_1\} \) if \( t = 1 \), and \( F_1 \) and \( F_2 \) are disjoint if \( t \geq 2 \). Define \( G \) to be the graph obtained from \( H \) by the addition of these internally disjoint arcs:

\[
\begin{align*}
A_2(x_{i+1}, y_{i+1}) & \quad (1 \leq i \leq t-1); \\
A_2(x_i, x_{i+1}) & \quad (1 \leq i \leq t-1); \\
A_3(x_i, x_{i+1}) & \quad (1 \leq i \leq t-1); \\
A_2(y_i, y_{i+1}) & \quad (1 \leq i \leq t-1); \\
A_3(y_i, y_{i+1}) & \quad (1 \leq i \leq t-1).
\end{align*}
\]

Thus, \(|V(G)| = 9t + 12\), and the only three vertices of degree 4 in \( G \) join 5-cycles in \( F_1 \cup F_2 \). Since every incremental subgraph has a vertex of degree 4, \( F_1 \) and \( F_2 \) are the only incremental subgraphs in \( G \). Since \( G \) can be obtained from \( R \) by repeated applications of operation 1 (or 3) of Theorem 3, \( G \) is \( C_5 \)-critical and \( K_4 \)-free.

REFERENCES


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