CONTRACTIONS OF GRAPHS WITH NO SPANNING EULERIAN SUBGRAPHS

P. A. CATLIN

Received March 10, 1986
Revised February 9, 1987

Let \( p \geq 2 \) be a fixed integer, and let \( G \) be a connected graph on \( n \) vertices. If \( \delta(G) \geq 2 \), if \( d(u) + d(v) > 2n/p - 2 \) holds whenever \( uv \notin E(G) \), and if \( n \) is sufficiently large compared to \( p \), then either \( G \) has a spanning eulerian subgraph, or \( G \) is contractible to a graph \( G_1 \) of order less than \( p \) and with no spanning eulerian subgraph. The case \( p = 2 \) was proved by Lesniak—Foster and Williamson. The case \( p = 5 \) was conjectured by Benhocine, Clark, Köhler, and Veldman, when they proved virtually the case \( p = 3 \). The inequality is best-possible.

1. Introduction

Consider a finite graph \( G \) with vertex set \( V(G) \) and edge set \( E(G) \). Let \( n \) denote the order of \( G \), and let \( G^c \) denote the complement of \( G \). Let \( d(v) \) denote the degree of \( v \) in \( G \), and let \( d_i(v) \) denote the degree of \( v \) in \( G_i \). The edge-connectivity of \( G \) is \( \kappa(G) \). Let \( a(G) \) denote the arboricity of \( G \); i.e., the minimum number of forests whose union contains \( E(G) \). We regard eulerian graphs as being connected, and a spanning eulerian subgraph of \( G \) is an eulerian subgraph containing every vertex of \( G \).

For \( xy \in E(G) \), an elementary contraction of \( G \) is the graph \( G / xy \) obtained from \( G \) by deleting \( \{x, y\} \) and inserting a new vertex \( v \) and edges joining \( v \) to each \( w \in V(G - \{x, y\}) \) with as many edges as \( \{x, y\} \) was joined to \( w \) by edges in \( G \). (Thus, an elementary contraction can create multiple edges). A contraction of \( G \) is a graph \( G/H \) obtained from \( G \) by a sequence of elementary contractions of edges of the subgraph \( H \).

Lesniak—Foster and Williamson [6] proved:

**Theorem 1.** Let \( G \) be a graph of order \( n \geq 6 \). If \( \delta(G) \geq 2 \) and if any pair \( u, v \) of non-adjacent vertices of \( G \),

\[
d(u) + d(v) \geq n - 1,
\]

then \( G \) has a spanning eulerian subgraph.

Benhocine, Clark, Köhler, and Veldman [1] recently proved:

**Theorem 2.** Let \( G \) be a 2-edge-connected graph on \( n \geq 3 \) vertices. If

\[
d(u) + d(v) \geq \frac{1}{3} (2n + 3)
\]

whenever \( uv \notin E(G) \), then \( G \) has a spanning eulerian subgraph.

AMS subject classification: 05 C 45.
In this paper, we shall generalize these results, using a new method. We first present a concept that we introduced in [2].

A graph $G$ is called collapsible if for any even set $S \subseteq V(G)$, there is a forest $\Gamma$ in $G$ such that both

i) $G - E(\Gamma)$ is connected; and

ii) $S$ is the set of vertices of odd degree in $\Gamma$.

We state some observations about collapsible graphs:

(2) The cycles $C_3$ and $C_4$ are collapsible.

Note that if $G$ is not 2-edge-connected, or if $G = C_k$ for $k \geq 4$, then $G$ is not collapsible. Also, $K_{2,t}$ is never collapsible, for any $t$. If $t \equiv 1$, then $K_t$ is collapsible, except when $t = 2$.

(3) If $H$ has two edge-disjoint spanning trees, then $H$ is collapsible.

Statement (3) follows from the fact that for any even subset $R$ of the vertices of a tree $T$, there is a forest $\Gamma$ in $T$ such that $R$ is the set of vertices of odd degree in $\Gamma$.

What makes the concept of collapsible graphs useful in the study of spanning eulerian subgraphs is the following proposition: Let $H$ be a connected subgraph of $G$. If $H$ is collapsible, then these are equivalent:

i) $G$ has a spanning eulerian subgraph;

ii) $G/H$ has a spanning eulerian subgraph.

Also, if $H$ is collapsible, then $G$ is collapsible iff $G/H$ is collapsible.

2. The main results

We prove our main result in terms of collapsible graphs, and in the corollaries we express it in terms of spanning eulerian subgraphs.

**Theorem 3.** Let $G$ be a connected simple graph of order $n$, and let $p \geq 2$ be an integer. If

(4) \[ d(u) + d(v) > \frac{2n}{p} - 2 \]

whenever $uv \in E(G)$, and if

(5) \[ n \equiv 4p^2, \]

then exactly one of the following conclusions holds:

a) $G$ is collapsible;

b) $G$ is contractible to a noncollapsible graph $G_1$ of arboricity $a(G_1) \equiv 2$ and of order less than $p$;

c) $p = 2$ and $G - x = K_{n-1}$ for some $x \in V(G)$ with $d(x) = 1$;

d) $p = 4$, and there is a contraction-mapping $G \rightarrow C_4$, such that the preimages of some adjacent pair of vertices of $C_4$ are adjacent singletons of degree 2 in $G$. Also, in every contraction of parts b) and d), the preimage of any vertex of $G_1$ is an induced collapsible subgraph of $G$.

First, we state some consequences of Theorem 3. We regard $K_4$ as having a spanning eulerian subgraph.
Corollary 1. Let $G$ be a connected simple graph of order $n$, and let $p \geq 2$ be an integer. If
\[ d(u) + d(v) > \frac{2n}{p} - 2 \]
whenever $uv \notin E(G)$, and if
\[ n \geq 4p^2, \]
then exactly one of the following conclusions holds:

a) $G$ has a spanning eulerian subgraph;

b) $G$ is contractible to a graph $G'$ of order less than $p$ and containing no spanning eulerian subgraph;

c) $p = 2$, and $G - x = K_{n-1}$ for some $x \in V(G)$ with $d(x) = 1$.

Corollary 2. Let $G$ be a 2-edge-connected simple graph of order $n$. If $n \geq 100$ and if
\[ d(u) + d(v) > \frac{2n}{5} - 2 \]
whenever $uv \notin E(G)$, then $L(G)$, the line graph of $G$, is hamiltonian, and $G$ has a spanning eulerian subgraph.

Corollary 2 (which is the case $p = 5$ of Corollary 1) is a conjecture of Benhocine, Clark, Köhler, and Veldman [1]. The case $p = 2$ of Corollary 1 is Theorem 1 (except for the bound on $n$), due to Lesniak—Foster and Williamson [6]. The case $p = 3$ of Corollary 1 is related to Theorem 2, which is a result of Benhocine, Clark, Köhler, and Veldman [1]. In Theorem 7 of [2], we proved a result related to the cases $p = 4$ of Theorem 3 and $p = 5$ of Corollary 1.

Proof of Corollary 1. Clearly, a) of Theorem 3 implies a) of Corollary 1. The same is true of c). Suppose, in b) and d) of Theorem 3, that the image $G_1$ of the contraction-mapping $G \rightarrow G_1$ has a spanning eulerian subgraph. Since the preimage of each vertex of $G_1$ is collapsible, it follows easily from the definition of collapsible graphs that $G$ has a spanning eulerian subgraph. If, in b) of Theorem 1, the contraction $G_1$ has no spanning eulerian subgraph, then neither does $G$.

Proof of Corollary 2. Set $p = 5$ in Corollary 1. Harary and Nash—Williams [4] showed that $G$ has a closed trail containing at least one end of each edge of $G$ iff $L(G)$ is hamiltonian.

Theorem 3 also can be applied to show that $G$ has a spanning $(x, y)$-trail, for every choice of $x, y \in V(G)$. For this conclusion to hold, the hypothesis of Corollary 2 is not sufficient when $G$ satisfies d) of Theorem 3. It would suffice if (8) is replaced by
\[ d(u) + d(v) > \frac{n}{2}. \]
This is best possible.

Corollary 3. Let $G$ be a 3-edge-connected simple graph of order $n$. If $n$ is sufficiently large and if
\[ d(u) + d(v) > \frac{n}{5} - 2 \]
whenever $uv \notin E(G)$, then $G$ has a spanning eulerian subgraph.
Proof. Set \( p = 10 \) in Corollary 1. If a) fails, then b) holds. By the definition of contractions,

\[ x'(G_1) \cong x'(G), \]

and so \( G_1 \) is 3-edge-connected. By inspection, there is no 3-edge-connected graph of order less than \( p \) with no spanning eulerian subgraph. Therefore, b) cannot hold. \( \square \)

Jaeger [5] showed that a graph containing two edge-disjoint spanning trees has a spanning eulerian subgraph (such a graph is also collapsible, by (3)). We have also used this method of collapsible graphs in another paper [3], to obtain other conditions for a graph to have a spanning eulerian subgraph.

Let \( G_1 \) be a graph of order \( p \) satisfying

i) \( G_1 \) has no spanning eulerian subgraph; and

ii) Any contraction \( G' \) of \( G_1 \) has a spanning eulerian subgraph.

The only such graphs \( G_1 \) of order at most 7 are \( K_2, K_{2,3}, K_{2,5}, \) and \( Q_3 - v \) (the cube minus a vertex).

We claim that for any \( p \geq 7 \), there is a graph \( G_1 \) of order \( p \) satisfying both i) and ii). When \( p \) is odd, \( G_1 = K_{2, p-3} \) is such a graph. We shall construct examples for even values of \( p \), next. Let \( H \) be a path of length 3 with consecutive vertices labelled \( x_1, x_2, x_3, x_4 \). Define the graph \( G(s, t) \) of order \( 4 + s + t \), to be the graph obtained from \( H \) by adding \( s \) vertices with neighbourhood \( \{x_1, x_3\} \) and \( t \) vertices with neighbourhood \( \{x_2, x_4\} \). Suppose \( s \) and \( t \) are even. Then the set \( S \) of odd-degree vertices of \( G(s, t) \) is \( S = \{x_1, x_3\} \). Because of the set \( S \), \( G(s, t) \) is not collapsible, for if \( \Gamma \) is a forest in \( G(s, t) \), with \( S \) as the set of odd-degree vertices of \( \Gamma \), then \( G(s, t) - E(\Gamma) \) is not connected. Therefore, if \( s \) and \( t \) are even, then \( G(s, t) \) has no spanning eulerian subgraph, and so for any even integer \( p \geq 8 \), \( G_1 = G(2, p - 6) \) satisfies condition i) above. Since \( G_1 \) also satisfies ii), our claim is true.

We shall now show that the inequalities (4), (6), (8), and (9) are best-possible.

Form the graph \( G \) by replacing each vertex of \( G_1 \) with a clique \( K_s \) \( (s \geq 1) \), such that the edges of \( E(G_1) \) join the corresponding cliques in \( G \), and so that \( G \) has order \( n = ps \) and is contractible to \( G_1 \), of order \( p \). Since \( G_1 \) has no spanning eulerian subgraph, neither has \( G \), and neither \( G_1 \) nor \( G \) is collapsible. Whenever \( u \notin E(G) \),

\[ d(u) + d(v) \leq \frac{2n}{p} - 2, \]

and if \( s > A(G_1) \), then equality holds for some \( u, v \in V(G) \). Thus, (4) and (6) barely fail, and the conclusions of Theorem 3 and Corollary 1 fail. When \( G_1 = K_{2,3} \), the corresponding \( G \) shows that (8) of Corollary 2 is best-possible, and when \( G_1 \) is the Petersen graph, the corresponding \( G \) shows that (9) of Corollary 3 is best-possible.

Corollary 3 holds even when its conclusion is changed to "\( G \) is collapsible". With a longer argument, it is possible to improve (5) and (7) to \( n \geq p^2 \), except for the following cases:

\[
\begin{align*}
p = 2, \quad n = 5, \quad G &= G_1 = K_{2,3}; \\
p = 5, \quad n \leq 32, \quad G_1 &= K_{2,3}; \\
p = 6, \quad n \leq 38, \quad G_1 &= \text{the bipartite theta graph of order 6.}
\end{align*}
\]
The first exceptional case arises in Theorem 1. In the latter two exceptional cases, as in d) of Theorem 3, there are two adjacent vertices \( x, y \in V(G_1) \), such that \( d_1(x) + d_1(y) = p \) and the preimages of \( x \) and \( y \) in \( G \) are singletons with \( d(x) + d(y) = p \) in \( G \). It also appears possible that even \( n \geq p^2 \) is not quite best-possible, but it is close. The details are tedious, and we omit them.

If the proof that follows were a proof of Corollary 1 directly, we would still define \( G_1 \) exactly as in the beginning of the proof that follows, in terms of contractions of collapsible subgraphs of \( G \).

3. The proof

The conclusions a), b), c), and d) of Theorem 3 are mutually exclusive.

Let \( G \) be a connected simple graph satisfying (4) and (5), but not a) of Theorem 3. Let \( E \subseteq E(G) \) be a minimal edge-set such that every component \( H_1, H_2, \ldots, H_c \) of \( G - E \) is collapsible. Since a) fails, \( G \) is not collapsible, and since each component \( K_i \) of \( G - E(G) \) is collapsible, \( E \) exists. If \( G \) has a cut-edge and \( p > 2 \) then let \( G_1 \) be a \( K_p \) (note that (b) is satisfied and we are done); but if \( G \) has no cut-edge or if \( p = 2 \), then let \( G_1 \) denote the graph obtained from \( G \) by contracting all edges of \( E(G) - E \). Since \( \omega(G - E) = c \),

\[
(10) \quad c = |V(G_1)|.
\]

By the minimality of \( E \), and by (2),

\[
(11) \quad G_1 \text{ has no 3-cycle; and}
\]

\[
(12) \quad G_1 \text{ has no multiple edges.}
\]

Since \( |E(G_1)| \geq 2c - 2 \) implies that some nontrivial induced subgraph \( H \) of \( G_1 \) contains two edge-disjoint spanning trees, both \( a(G_1) \geq 2 \) and

\[
(13) \quad |E| = E(G_1)| \leq 2c - 3
\]

follow from (3), the minimality of \( E \), and (10). (By results we obtained in [2], either \( G_1 \) has a bridge or the inequality (13) is strict.)

If \( G \) has a cut-edge, then since \( G \) is simple, is straightforward to show that (4) of Theorem 3 implies either conclusion b) \( (p > 2; G_1 \text{ has a cut-edge}) \) or conclusion c) \( (p = 2) \). Hence, we shall suppose \( \chi'(G) \geq 2 \) and hence that

\[
(14) \quad \chi'(G_1) \geq 2.
\]

Since the smallest 2-edge-connected noncollapsible graph is \( G_1 = C_4 \), we may suppose, without loss of generality, that

\[
(15) \quad c \equiv 4.
\]

We shall use the following lemma:

**Lemma.** Let \( H \) be a graph, and for each \( x \in V(H) \), define

\[
B(x) = \{w \in V(H) | wx \in E(H^c)\}
\]

If \( H \) is triangle-free, and not a star, then the family \( (B(x) | x \in V(H)) \) has a complete system of distinct representatives.
**Proof.** Let $H$ be triangle-free and not a star. If $H$ is the five-cycle, then the lemma holds. We claim that if $H \neq C_5$, then $H^c$ has a spanning subgraph in which each component is either $K_2$ or $K_3$. Note that the lemma follows, if we prove this claim.

By way of contradiction, suppose

(16) 
$H$ is a smallest counterexample to the claim.

By inspection, we may suppose

(17) 
$|V(H)| \geq 6$.

Since $H$ is triangle-free and since the Ramsey number $r(3, 3)$ is 6, (17) implies that $H$ contains an independent set $\{x, y, z\} \subseteq V(H)$.

If $H - \{x, y, z\}$ is not a star, then by (16), $H - \{x, y, z\}$ satisfies the claim. Since $H^c[\{x, y, z\}] = K_3$, $H$ satisfies the claim, contrary to (16).

If $H - \{x, y, z\}$ is a star, then by (17), $H - \{x, y\}$ is not a star. By (16), $H - \{x, y\}$ satisfies the claim, and since $H^c[\{x, y\}] = K_2$, $H$ satisfies the claim, contrary to (16).

**Proof of Theorem 3, continued.** Since a) fails for $G$, a) also fails for $G_1$ (see [2], Theorem 6). This satisfies a requirement of b). Let

$V(G_1) = \{x_1, x_2, \ldots, x_c\}$.

By (11) and (14), $G_1$ satisfies the hypotheses of the lemma. Therefore, there is a system of distinct representatives $y_i \in B(x_i)$ ($1 \leq i \leq c$). The resulting set of ordered pairs

$\{(x_1, y_1), (x_2, y_2), \ldots, (x_c, y_c)\}$

corresponds to a set $E''$ (with multiplicities allowed) of $c$ edges of $G_1^c$:

$E'' = \{x_1y_1, x_2y_2, \ldots, x_cy_c\}$.

(A multiplicity occurs when $x_i = y_j$ and $x_j = y_i$ for some $i$ and $j$.) Let

$G'' = G_1^c[E'']$ 

be the subgraph of $G_1^c$ induced by $E''$. Clearly,

(18) 
$G''$ is a spanning subgraph of $G_1^c$;

(19) 
Each component of $G''$ is either a $K_2$ or a cycle; and

(20) An edge occurs more than once in $E''$ iff it occurs exactly twice, and is the edge of a $K_2$ component of $G''$.

Let $\Theta: G \rightarrow G_1$ denote the contraction-mapping defining $G_1$. For each edge $xy \in E(G)^c \subseteq E(G_1^c)$, the preimages $\Theta^{-1}(x)$ and $\Theta^{-1}(y)$ are distinct components of $G - E$, with no edge of $G$ joining a vertex of $\Theta^{-1}(x)$ to a vertex of $\Theta^{-1}(y)$. For all $i$ with $1 \leq i \leq c$, pick $u_i \in \Theta^{-1}(x_i)$ and $v_i \in \Theta^{-1}(y_i)$. Then $u_iv_i \in E(G'')$, for all $i$. Denote by $E'$ the set (possible with multiplicities)

$E' = \{u_1v_1, u_2v_2, \ldots, u_cv_c\}$.

Hence, $\Theta[E'] = E''$, and by (18), each component of $G - E$ contains one member of $U = \{u_1, u_2, \ldots, u_c\}$; and since $\{y_1, y_2, \ldots, y_c\}$ is a transversal, each component of $G - E$ contains one member of $V = \{v_1, v_2, \ldots, v_c\}$. 


Define $N_{G-E}(x)$ to be the neighbourhood of $x$ in $G-E$. If for some $j$ and $k$, $u_i \in V(H_j)$ and $v_i \in V(H_k)$, then
\begin{equation}
|N_{G-E}(u_i)| + |N_{G-E}(v_i)| \equiv |V(H_j)| - 1 + |V(H_k)| - 1.
\end{equation}
Since each component of $G-E$ contains exactly one $u_i \in U$ and one $v_i \in V$, we can sum (21) over $E'$ and get
\begin{equation}
\sum_{i=1}^{c} (|N_{G-E}(u_i)| + |N_{G-E}(v_i)|) \equiv \\
\equiv \sum_{j=1}^{c} |V(H_j)| - 1 + \sum_{k=1}^{c} |V(H_k)| - 1 = 2(n-c).
\end{equation}
By $E = E(G_1)$ and by (13), there are at most $2|E| \leq 4|V(G_1)| - 6 = 4c - 6$ incidences in $G$ of edges of $E$ with $U$ and at most $2|E| \leq 4c - 6$ incidences in $G$ of edges of $E$ with $V$. Hence,
\begin{equation}
\sum_{i=1}^{c} d(u_i) + d(v_i) = 4|E| + \sum_{i=1}^{c} (|N_{G-E}(u_i)| + |N_{G-E}(v_i)|) \equiv \\
\equiv 2(4c - 6) + 2(n-c) = 2n + 6c - 12.
\end{equation}
Finally, we are ready to use the hypothesis of Theorem 3. Since $u_i v_i \in E(G^c)$, (4) and (22) give
\begin{equation}
c \left( \frac{2n}{p} - 2 \right) \leq \sum_{i=1}^{c} d(u_i) + d(v_i) \leq 2n + 6c - 12
\end{equation}
\begin{equation}
c(n-4p) \leq np - 6p
\end{equation}
\begin{equation}
c \leq \frac{np - 6p}{n-4p}
\end{equation}
which is less than $p+1$, by (5). Hence, by (10),
\begin{equation}
|V(G_j)| = c \equiv p
\end{equation}
Suppose that (24) holds with equality; i.e., suppose
\begin{equation}
c = p
\end{equation}
We then show that we have case d).

Arrange the components $H_1, H_2, \ldots, H_c$ of $G-E$ such that
\begin{equation}
|V(H_1)| \equiv |V(H_2)| \equiv \ldots \equiv |V(H_c)|.
\end{equation}
Case I. Suppose $|V(H_j)| > \Delta(G_1)$.

Then in (22) we can choose $u_i$ and $v_i$, for $1 \leq i \leq c$, so that they are not incident with $E$, and so the $4|E|$ term disappears from (22):
\begin{equation}
\sum_{i=1}^{c} d(u_i) + d(v_i) \leq 2(n-c).
\end{equation}
By (4) and (27),
\[(28)\quad c \left( \frac{2n}{c} - 2 \right) < 2(n - c),\]
a contradiction. Therefore,
\[(29)\quad |V(H_i)| \leq \Delta(G_1) \leq c - 1,\]
and the latter inequality follows from (12).

**Case II.** Suppose that for every \(i \geq 2\), there is an \(x_1 \in V(H_i)\) and an \(x_i \in V(H_i)\) such that \(x_1 x_i \in E(G)\). By (4), (25), (29), and (13), for \(i \geq 2\),
\[\tag{30} \frac{2n}{c} - 2 < d(x_1) + d(x_i) \equiv\]
\[\equiv E| + |V(H_i)| + |V(H_i)| - 2 \equiv 3c - 4 + |V(H_i)| - 2.\]
We sum (30) over all \(i \geq 2\) to get
\[\left( c - 1 \right) \frac{2n}{c} < (3c - 4)(c - 1) + \sum_{i=1}^{c} |V(H_i)| =\]
\[< (3c - 4)(c - 1) + n,\]
which, by (15), is false for large \(n\).

**Case III.** Suppose that for some \(k \geq 2\), \(x_1 x_k \in E(G)\) for all \(x_i \in V(H_i)\) and \(x_k \in V(H_k)\). Since \(G_1\) is simple, by (12), this implies
\[\tag{31} |V(H_1)| = |V(H_k)| = 1.\]
Suppose \(k \geq 3\). Then (31) and (26) imply \(V(H_i) = \{x_i\}\) for \(1 \leq i \leq 3\). By (11), two of \(\{x_1, x_2, x_3\}\) are not adjacent in \(G\), say \(x_1\) and \(x_2\). Then by (4), (25), and (13),
\[\frac{2n}{c} - 2 < d(x_1) + d(x_2) \leq |E| \leq 2c - 3,\]
which contradicts (5). Hence, \(k = 2\) and
\[\tag{32} |V(H_j)| > 1,\]
if \(j \geq 3\). By (12),
\[\tag{33} d(x_i) \leq c - 1,\]
and at most one edge of \(E(G)\) joins \(V(H_1)\) and \(V(H_j)\) \((3 \leq j \leq c)\). Thus by (32) there is an \(x_j \in V(H_j) - N(x_1)\) whenever \(3 \leq j \leq c\), and so
\[\tag{34} \sum_{j=3}^{c} d(x_j) \equiv (2|E| - 1) + \sum_{j=1}^{c} (|V(H_j)| - 1) \equiv (2|E| - 1) + n - c.\]
By (4), (13), (25), (33), and (34),
\[\tag{35} (c - 2) \frac{2n}{c} < \sum_{j=3}^{c} d(x_j) \leq (c - 2) d(x_1) + d(x_j) \equiv (c - 2) d(x_1) + (n - c + 2|E| - 1) \equiv\]
\[\equiv (c - 2)(c - 1) + n + 3c - 7.\]
By (15), \( c \geq 4 \). Unless \( c = 4 \), (35), (5), and (15) combine to give a contradiction. When \( c = 4 \), (25) and (31) and \( k = 2 \) imply that d) of Theorem 3 holds. This completes Case III and the proof of Theorem 3. □

References


Paul A. Catlin

Wayne State University
Detroit, Michigan 48202, U.S.A.