Nowhere-zero 3-flows in triangularly connected graphs

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ABSTRACT. Let H_1 and H_2 be two subgraphs of a graph G. We say that G is the 2-sum of H_1 and H_2, denoted by $H_1 \oplus_2 H_2$, if $E(H_1) \cup E(H_2) = E(G)$, $|V(H_1) \cap V(H_2)| = 2$, and $|E(H_1) \cap E(H_2)| = 1$. A triangle-path in a graph G is a sequence of distinct triangles $T_1T_2 \cdots T_m$ in G such that for $1 \leq i \leq m-1$, $|E(T_i) \cap E(T_{i+1})| = 1$ and $E(T_i) \cap E(T_j) = \emptyset$ if $j > i+1$. A connected graph G is triangularly connected if for any two edges e and e', which are not parallel, there is a triangle-path $T_1T_2 \cdots T_m$ such that $e \in E(T_1)$ and $e' \in E(T_m)$. Let G be a triangularly connected graph with at least three vertices. We prove that G has no nowhere-zero 3-flow if and only if there is an odd wheel W and a subgraph G_1 such that $G = W \oplus_2 G_1$, where G_1 is a triangularly connected graph without nowhere-zero 3-flow. Repeatedly applying the result, we have a complete characterization of triangularly connected graphs which have no nowhere-zero 3-flow. As a consequence, G has a nowhere-zero 3-flow if it contains at most three 3-cuts. This verifies Tutte's 3-flow Conjecture and an equivalent version by Kochol for triangularly connected graphs. By the characterization, we obtain extensions to earlier results on locally connected graphs, chordal graphs and squares of graphs. As a corollary, we obtain a result of Barát and Thomassen that every triangulation of a surface admits all generalized Tutte-orientations.

Key Words: Nowhere-zero flows, triangularly connected, locally connected, 2-sum of graphs, chordal graphs, squares of graphs.

1 Introduction

The graphs considered here may have parallel edges, but no loops. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The *degree* of a vertex v, denoted by $d(v)$, is the number of edges incident with v. For $xy \in E(G)$, we call y a *neighbor* of x, and the set of neighbors of x is denoted by $N_G(x)$, or simply $N(x)$. An edge is *contracted* if it is deleted and its two ends are identified into a single vertex. Let H be a connected subgraph of G. G/H denotes the graph obtained from G by contracting all the edges of H and deleting all the resulting loops. An *edge-cut* (*vertex-cut*) is a set of edges (vertices) whose removal leaves a graph with more components. A connected graph is *k-edge-connected* (*k-connected*) if it has no edge-cut (vertex-cut) of ℓ edges (vertices) for any $\ell < k$. For simplicity, an edge-cut of k edges is called a *k-cut*.

A *k-circuit* is a circuit of k edges. A *wheel* W_k is the graph obtained from a k-circuit by adding a new vertex, called the *center* of the wheel, which is joined to every vertex of the k-circuit. W_k is an *odd (even) wheel* if k is odd (even). For a technical reason, a single edge is regarded as 1-circuit, and thus W_1 is a triangle, called the *trivial wheel.*

Let G be a graph with an orientation. For each vertex $v \in V(G)$, $E^+(v)$ is the set of non-loop edges with tail v, and $E^-(v)$ is the set of non-loop edges with head v. Let $\mathbb{Z}_k$ denote an abelian group of k elements with identity 0. Let f be a function from $E(G)$ to $\mathbb{Z}_k$. Set

$$f(v) = \sum_{e\in E^+(v)} f(e) - \sum_{e\in E^-(v)} f(e).$$

f is called a *$\mathbb{Z}_k$-flow* in G if $f(v) = 0$ for each vertex $v \in V(G)$. The *support* of f is defined by $S(f) = \{e \in E(G) : f(e) \neq 0\}$. f is *nowhere-zero* if $S(f) = E(G)$. It is well known that a graph G has a nowhere-zero $\mathbb{Z}_k$-flow if and only if there is an integer-valued function f on $E(G)$ such that $0 < |f(e)| < k$ for each $e \in E(G)$, and $f(v) = 0$ for each $v \in V(G)$, which is called a *nowhere-zero k-flow* in G. Therefore, we also call a $\mathbb{Z}_k$-flow a *k-flow.* We shall restrict our attention to the case that $k = 3$. Since loops play no role with respect to existence of nowhere-zero flows, we only consider loopless graphs. The well-known 3-flow conjecture of Tutte (see unsolved problem 48 of [1]) is that

Conjecture 1.1 *Every 4-edge-connected graph has a nowhere-zero 3-flow.*

Jaeger et al. [4] introduced the property of $\mathbb{Z}_k$-connectedness, which can be regarded as an extension of $\mathbb{Z}_k$-flows. A graph G is *$\mathbb{Z}_k$-connected* if for any function $b : V(G) \to \mathbb{Z}_k$ with $\sum_{v\in V(G)} b(v) = 0$, G has a nowhere-zero $\mathbb{Z}_k$-flow f such that $f(v) = b(v)$ for each vertex $v \in V(G)$. Clearly, if G is $\mathbb{Z}_k$-connected, then it has a nowhere-zero k-flow (with

$b = 0$). But, the converse is not true. An n-circuit ($n \geq 2$) has a nowhere-zero k-flow for any $k \geq 2$, but it is not $\mathbb{Z}_k$-connected if $k \leq n$. In the same paper, Jaeger et al. [4] constructed a 4-edge-connected graph which is not $\mathbb{Z}_3$-connected, and conjectured that every 5-edge-connected graph is $\mathbb{Z}_3$-connected. By a result of Kochol [6], the truth of this conjecture would imply the truth of Conjecture 1.1 above.

In this paper, instead of the $\mathbb{Z}_k$-connectedness, we use the concept of k-flow contractibility. A connected graph H is *k-flow contractible* if for any graph G with H as a subgraph (it is allowed that $G = H$), any nowhere-zero $\mathbb{Z}_k$-flow f in G/H can be extended to a nowhere-zero $\mathbb{Z}_k$-flow g in G such that f is the restriction of g on $E(G/H)$. This definition is different from the usual one, which does not require that f is the restriction of g on $E(G/H)$. Thus, by our definition, the 4-circuit C_4 is not 4-flow contractible, while a nowhere-zero 4-flow in G/C_4 is indeed extendible to a nowhere-zero 4-flow in G . The following proposition shows that the k-flow contractibility is equivalent to the $\mathbb{Z}_k$-connectedness.

Proposition 1.2 *A graph is $\mathbb{Z}_k$-connected if and only if it is k-flow contractible.*

Proof. Suppose that H is a $\mathbb{Z}_k$-connected graph. We shall prove that H is k-flow contractible. For any graph G with H as a subgraph, let f be a nowhere-zero $\mathbb{Z}_k$-flow in G/H. Consider each vertex $v \in V(H)$ as a vertex in G and f as a function on $E(G)$ with $f(e) = 0$ for each $e \in E(H)$. Define a function b: $V(H) \to \mathbb{Z}_k$ by

$$b(v) = f(v) \quad \text{for each} \quad v \in V(H).$$

Since H is $\mathbb{Z}_k$-connected, there is a nowhere-zero function h: $E(H) \to \mathbb{Z}_k$ such that $h(v) = -b(v)$ for each $v \in V(H)$. Consider h as a function on $E(G)$ by setting $h(e) = 0$ if $e \in E(G) \setminus E(H)$. Then, $f + h$ is a nowhere-zero $\mathbb{Z}_k$-flow in G, whose restriction on $E(G/H)$ is f.

Conversely, suppose that H is k-flow contractible. For any function b: $V(H) \to \mathbb{Z}_k$ with $\sum_{v \in V(H)} b(v) = 0$, let $X = \{x_i \in V(H) : b(x_i) \neq 0\}$, and let G be the graph obtained from H by adding a new vertex joined to each vertex $x_i \in X$ by an edge e_i, $1 \leq i \leq |X|$. By the definition of k-flow contractibility, H is connected, and so G/H has a nowhere-zero $\mathbb{Z}_k$-flow f given by $f(e_i) = b(x_i)$, $1 \leq i \leq |X|$. (If $X = \emptyset$, then $G = H$ and $f = \emptyset$.) Since H is k-flow contractible, f can be extended to a nowhere-zero $\mathbb{Z}_k$-flow g in G with $g(e_i) = f(e_i)$, $1 \leq i \leq |X|$. Let h be the restriction of g on $E(H)$. Then, $h(v) = b(v)$ for each $v \in V(H)$, and hence H is $\mathbb{Z}_k$-connected. ■

The following is a basic property of k-flow contractible graphs and was proved in [9] and [3]. We present here an alternative proof.

Observation 1.3 *Let H be a k-flow contractible subgraph in G. If G/H is k-flow contractible, then so is G.*

Proof. Let G' be a graph with G as a subgraph, and f' a nowhere-zero $\mathbb{Z}_k$-flow in G'/G. Let $F = G/H$, and consider f' as a nowhere-zero $\mathbb{Z}_k$-flow in G'/F $(= G'/G)$. Since F is k-flow contractible, we can extend f' to a nowhere-zero $\mathbb{Z}_k$-flow f in G'/H, but H is also k-flow contractible, and thus f, and so f', can be extended to a nowhere-zero $\mathbb{Z}_k$-flow in G', which shows that G is k-flow contractible. ■

Let H_1 and H_2 be two subgraphs of a graph G. We say that G is the *k-sum* of H_1 and H_2, denoted by $H_1 \oplus_k H_2$, if $E(H_1) \cup E(H_2) = E(G)$, $|V(H_1) \cap V(H_2)| = k$, and $E(H_1) \cap E(H_2)$ is the complete graph on k vertices. In this paper, we restrict our attention to 2-sum.

Let G be a graph. A *triangle-path* in G is a sequence of distinct triangles $T_1T_2 \cdots T_m$ in G such that for $1 \le i \le m-1$,

$$|E(T_i) \cap E(T_{i+1})| = 1 \quad \text{and} \quad E(T_i) \cap E(T_j) = \emptyset \quad \text{if } j > i+1. \tag{1.1}$$

Furthermore, if $m \ge 3$ and (1.1) holds for all i, $1 \le i \le m$, with the addition taken mod m, then the sequence is called a *triangle-cycle*. The number m is the *length* of the triangle-path (triangle-cycle). A connected graph G is *triangularly connected* if for any distinct $e, e' \in E(G)$, which are not parallel, there is a triangle-path $T_1T_2 \cdots T_m$ such that $e \in E(T_1)$ and $e' \in E(T_m)$. Trivially, a single edge is triangularly connected. This is technically useful for the representations of the following main theorems, giving complete characterizations of triangularly connected graphs which are not 3-flow contractible or have no nowhere-zero 3-flow.

The main result of this paper is the following characterization of triangularly connected graphs which are not 3-flow contractible.

Theorem 1.4 *Let G be a triangularly connected graph with $|V(G)| \ge 3$. Then G is not 3-flow contractible if and only if there is an odd wheel W and a subgraph G_1 such that $G = W \oplus_2 G_1$, where G_1 is triangularly connected and not 3-flow contractible.*

In the above characterization, odd wheels include the trivial odd wheel (triangle). By excluding the trivial odd wheel, we have the following characterization of triangularly connected graphs which have no nowhere-zero 3-flow.

Theorem 1.5 *Let G be a triangularly connected graph with $|V(G)| \geq 3$. Then G has no nowhere-zero 3-flow if and only if there is a nontrivial odd wheel W and a subgraph G_1 such that $G = W \oplus_2 G_1$, where G_1 is a triangularly connected graph without nowhere-zero 3-flow.*

A classical result on graph coloring is Grötzsch's theorem: every 2-edge-connected planar graph without triangle is 3-vertex-colorable (see [10]), which was extended to 2-edge-connected planar graphs with at most three triangles by Grünbaum and Aksionov (see [5], page 7). A facial triangle in a plane graph corresponds to 3-cut in its dual. A plane graph has a nowhere-zero 3-flow if and only if it is 3-face-colorable. Hence, the dual version of the above-mentioned theorem is that every 2-edge-connected planar graph with at most three 3-cuts has a nowhere-zero 3-flow. It is interesting to note that Kochol [7] has proved that Conjecture 1.1 is equivalent to: every 2-edge-connected graph with at most three 3-cuts has a nowhere-zero 3-flow. In Section 4, as an application of Theorem 1.5, we obtain the following corollary, which verifies Conjecture 1.1 and the equivalent version by Kochol [7] for triangularly connected graphs.

Corollary 1.6 *Let G be a triangularly connected graph with $|V(G)| \geq 3$. If G contains at most three 3-cuts, then G has a nowhere-zero 3-flow.*

As seen in Proposition 1.2, "3-flow contractible" is equivalent to "$\mathbb{Z}_3$-connected", which is the same as "admitting all generalized Tutte-orientations" defined by Barát and Thomassen [2]. Thus, Theorem 4.5 in [2] can be restated as:

Corollary 1.7 *Let G be a triangulation of a surface. Then G is 3-flow contractible, unless $G = K_4$ or K_3.*

Proof. Clearly, G is triangularly connected and $|V(G)| \geq 3$. Since $G \neq K_4$ or K_3, it cannot be the 2-sum of an odd wheel and a 3-flow non-contractible graph. The corollary follows from Theorem 1.4. ■

2 3-flow contractible graphs

In this section, we study graphs which are 3-flow contractible. For the sake of simplicity, throughout this section we abbreviate "3-flow contractible" to "*contractible*". A graph is *non-contractible* if it is not 3-flow contractible. It is clear that a 2-circuit is contractible. However, a triangle is non-contractible, as demonstrated by K_4, the complete graph on 4 vertices. It was first observed by Lai [8] that the 2-sum of two non-contractible graphs is non-contractible. A full version of this was given by DeVos et al. [3].

Lemma 2.1 [3, Proposition 2.5] *Let $G = H_1 \oplus_2 H_2$.*

(i) If neither H_1 nor H_2 has a nowhere-zero 3-flow, then G does not have a nowhere-zero 3-flow.

(ii) If neither H_1 nor H_2 is contractible, then G is not contractible.

The following is a technical lemma, which describes a useful property of triangularly connected graphs. Roughly speaking, the lemma tells that in a triangularly connected graph, an edge of zero flow-value can be "switched" to any place in the graph.

Lemma 2.2 *Let f be a $\mathbb{Z}_3$-flow in a graph G and $H = T_1T_2 \cdots T_m$ a triangle-path in G. Suppose that g is another $\mathbb{Z}_3$-flow in G such that the restrictions of g and f on $E(G/H)$ are identical, and subject to this, $|S(g) \cap E(H)|$ is maximum. Then,*

(i) If there is an edge $a \in E(H)$ with $g(a) = 0$, then for any edge $w \in E(H)$, there is a $\mathbb{Z}_3$-flow ϕ in G such that $S(\phi) = (S(g) \setminus \{w\}) \cup \{a\}$.

(ii) There is at most one edge $a \in E(H)$ with $g(a) = 0$.

Proof. (i) Let $a \in E(T_p)$ and $w \in E(T_q)$. We may assume $q \geq p$, and choose p, q such that $q - p$ is as small as possible. Let $E(T_p) = \{a, a', a''\}$, where $a' \in E(T_p) \cap E(T_{p+1})$. Choose an orientation in G such that T_p is a directed triangle and let φ be a $\mathbb{Z}_2$-flow in G with $S(\varphi) = E(T_p)$. If $\{g(a'), g(a'')\} \neq \mathbb{Z}_3 \setminus \{0\}$, then there is $\alpha \in \mathbb{Z}_3 \setminus \{0\}$ such that $\phi = g + \alpha \cdot \varphi$ is a $\mathbb{Z}_3$-flow in G with $|S(\phi) \cap E(H)| > |S(g) \cap E(H)|$, a contradiction. Thus, $\{g(a'), g(a'')\} = \mathbb{Z}_3 \setminus \{0\}$. If $w \in \{a', a''\}$, let $\phi = g - g(w) \cdot \varphi$, and then $S(\phi) = (S(g) \setminus \{w\}) \cup \{a\}$, as required. Therefore, suppose that $q > p$. Let $\psi = g + g(a'') \cdot \varphi$. Then, $S(\psi) = (S(g) \setminus \{a'\}) \cup \{a\}$. Now, $\psi(a') = 0$ and $a' \in E(T_{p+1})$. Repeat the arguments to T_{p+1}. Eventually, we have a $\mathbb{Z}_3$-flow ϕ in G such that $S(\phi) = (S(g) \setminus \{w\}) \cup \{a\}$, as required by (i).

(ii) Suppose, to the contrary, that a_1 and a_2 are two distinct edges in H with $g(a_1) = g(a_2) = 0$. Let $a_1 \in E(T_p)$ and $a_2 \in E(T_q)$. We may assume that $q \geq p$ and $q - p$ is as small as possible. From (i), we see that a_1 and a_2 can not be in the same triangle, that is, $q \geq p + 1$. Let w be the edge of $E(T_{q-1}) \cap E(T_q)$. By (i), there is a $\mathbb{Z}_3$-flow ϕ in G such that $S(\phi) = (S(g) \setminus \{w\}) \cup \{a_1\}$. Now, $\phi(w) = 0$, and w, a_2 are in the same triangle T_q, a contradiction. This proves Lemma 2.2. ■

Lemma 2.3 *Let $H = T_1T_2 \cdots T_m$ be a triangle-path, where $m \geq 2$. Suppose that $V(T_1) = \{x, y, z\}$, where $\{yz\} = E(T_1) \cap E(T_2)$. If $x \in V(T_i)$ for some $i \neq 1$, then H is contractible.*

Proof. Let G be a graph containing H as a subgraph, and f a nowhere-zero $\mathbb{Z}_3$-flow in G/H. Let $H_1 = T_2T_3 \cdots T_m$ and $G' = G \setminus \{xy, xz\}$. Since $x \in V(H_1)$, $G/H = G'/H_1$, and so, f is a nowhere-zero $\mathbb{Z}_3$-flow in G'/H_1. Since H_1 is connected, we may extend f into a $\mathbb{Z}_3$-flow f' in G' such that the restriction of f' on $E(G'/H_1)$ is f. We may suppose that f' has been chosen so that $|S(f') \cap E(H_1)|$ is maximum. By Lemma 2.2, either f' is nowhere-zero or yz is the only edge with $f'(yz) = 0$. Let φ be a $\mathbb{Z}_2$-flow with $S(\varphi) = E(T_1)$. Let $\alpha = f'(yz)$ if $f'(yz) \neq 0$, and $\alpha \in \mathbb{Z}_3 \setminus \{0\}$ otherwise. Then $\phi = f' + \alpha \cdot \varphi$ is a nowhere-zero $\mathbb{Z}_3$-flow in G such that ϕ and f have the same restriction on $E(G/H)$. This shows that H is contractible. ■

Lemma 2.4 *Let G be a triangularly connected graph with $|V(G)| \geq 2$. Then G is contractible if and only if it contains a nontrivial contractible subgraph.*

Proof. If G is contractible, the statement is trivially true with the subgraph as G itself. Suppose that G has a nontrivial contractible subgraph H. Since G is triangularly connected, there is a sequence of graphs $G_1, G_2, \cdots, G_m$ such that $G_1 = G/H$, each G_{i+1} is obtained from G_i by contracting a 2-circuit, which is contractible, $1 \leq i \leq m-1$, and G_m is the trivial graph. By Observation 1.3, each G_i is contractible, $1 \leq i \leq m$, and so G is contractible. ■

Lemma 2.5 *Let H be a triangle-cycle. If H has two vertices of degree more than 3, then H is contractible.*

Proof. Let $H = T_1T_2 \cdots T_m$. Let $x, y \in V(H)$ be two distinct vertices of degree more than 3, say $x \in V(T_i)$ and $y \in V(T_j)$, where $1 \leq i \leq j \leq m$. Choose i and j such that $j - i$ is as small as possible. We claim that $j = i$, that is, T_i contains two vertices of degree more than 3. Let $V(T_i) = \{x, x', x''\}$. If $j > i$, then by the choice of i and j, $x \notin V(T_{i+1})$, and thus $\{x'x''\} = E(T_i) \cap E(T_{i+1})$, which implies, by the definition of triangle-paths, that $x \in V(T_{i-1})$. Without loss of generality, assume that $xx' \in E(T_{i-1}) \cap E(T_i)$, and so $d(x') > 3$, contradicting the choice of i and j. This shows that $j = i$, as claimed. Thus, we may let $y = x'$. By the definition of triangle-paths, either $x \in V(T_{i-1})$ or $x \in V(T_{i+1})$, say that $x \in V(T_{i-1})$.

(i) $xy \in E(T_{i-1}) \cap E(T_i)$. Let $H' = T_iT_{i+1} \cdots T_{i-2}$. We have that either $yx'' \in E(T_i) \cap E(T_{i+1})$ or $xx'' \in E(T_i) \cap E(T_{i+1})$. In the former, $x \in V(T_\ell)$ since $d(x) > 3$, and in the latter $y \in V(T_\ell)$ since $d(y) > 3$, for some triangle T_ℓ in H' with $\ell \neq i$. It follows from Lemma 2.3 that H' is contractible. But H' is a subgraph of H, which is triangularly connected, and therefore, by Lemma 2.4, H is contractible.

(ii) $xx'' \in E(T_{i-1}) \cap E(T_i)$. If $yx'' \in E(T_i) \cap E(T_{i+1})$, let $H' = T_iT_{i+1} \cdots T_{i-2}$, and as above, using $d(x) > 3$ we are done. If $xy \in E(T_i) \cap E(T_{i+1})$, then, replacing i by $i-1$, we have case (i) above. This completes the proof of Lemma 2.5. ■

Lemma 2.6 *A triangle-cycle is contractible if and only if it is not an odd wheel.*

Proof. Let H be a triangle-cycle. By the definition, H contains at least three triangles. If H is an odd wheel, then clearly H is non-contractible (in fact, H has no nowhere-zero $\mathbb{Z}_3$-flow). Suppose conversely that H is not an odd wheel. If H contains two vertices of degree more than 3, then by Lemma 2.5, we are done. Otherwise, H is an even wheel, and by a result of DeVos et al. [3, Proposition 2.4], H is contractible. ■

3 Proof of the main theorems

Proof of Theorem 1.4: If there are graphs W and G_1, as described in the theorem, such that $G = W \oplus_2 G_1$, then by Lemma 2.1(ii), G is not 3-flow contractible.

Conversely, suppose that G is not 3-flow contractible. We use induction on $|V(G)|$. If $|V(G)| = 3$, then G is a triangle, and the theorem trivially holds with $W = G$ and G_1 being a single edge. Suppose therefore that $|V(G)| \geq 4$ and the theorem holds for every triangularly connected graph G' with $|V(G')| < |V(G)|$.

Since G is triangularly connected and $|V(G)| \geq 3$, we see that G is 2-connected. If G has a vertex-cut consisting of two vertices, say x and y, then $xy \in E(G)$, and there are subgraphs G_1 and G_2 with $|V(G_i)| \geq 3$ $(i = 1, 2)$ such that $G = G_1 \oplus_2 G_2$. Choose x and y such that $V(G_1)$ is as small as possible, subject to $|V(G_1)| \geq 3$. If G_1 is an odd wheel, then by Lemma 2.4, G_2 is not 3-flow contractible, and we are done. Suppose thus that this is not the case. Clearly, G_1 is triangularly connected, and by Lemma 2.4, G_1 is not 3-flow contractible. By the induction hypothesis, there is an odd wheel W and a subgraph H of G_1 such that $G_1 = W \oplus_2 H$, where $|V(H)| \geq 3$ since G_1 is not an odd wheel. Let $x'y' \in E(W) \cap E(H)$. By the minimality of $|V(G_1)|$, $x'y' \neq xy$. By the definition of 2-sum, xy is entirely contained in W or H, in either case, $\{x', y'\}$ is a vertex-cut of G contradicting the choice of $\{x, y\}$. We suppose therefore that G is 3-connected, and shall show that

Claim. G contains a triangle-cycle.

Let $H = T_1T_2 \cdots T_m$ be a longest triangle-path in G, where $V(T_1) = \{x, y, z\}$ and $\{yz\} = E(T_1) \cap E(T_2)$. By Lemmas 2.3 and 2.4, $x \notin V(T_i)$ for all $i \geq 2$. Since G is 3-connected, $d(x) \geq 3$ and there is triangle T containing x such that $|E(T) \cap E(T_1)| \leq 1$.

(i) $|E(T) \cap E(T_1)| = 1$, say $E(T) \cap E(T_1) = \{xy\}$. Let w be the third vertex in T other than x and y. Since $x \notin V(T_i)$ for all $i \geq 2$, we have that $xw \notin E(H)$. If $yw \in E(T_s)$ for some s, then $TT_sT_{s-1} \cdots T_1$ is a triangle-cycle, as claimed. Otherwise, $yw \notin E(H)$, and then $TT_1T_2 \cdots T_m$ is a longer triangle-path in G, a contradiction.

(ii) $E(T) \cap E(T_1) = \emptyset$. Since G is triangularly connected, there is a triangle-path $Q = Q_1 Q_2 \cdots Q_t$ such that $Q_1 = T_1$ and $Q_t = T$. Let e be the edge of $E(Q_1) \cap E(Q_2)$. If $e \in \{xy, xz\}$, then we have (i) above with Q_2 in place of T. Suppose thus that $e = yz$. But, $x \in V(Q_t)$, it follows from Lemmas 2.3 and 2.4 that G is 3-flow contractible, a contradiction. This proves the claim.

By the claim above, let W be a triangle-cycle in G, and by Lemma 2.6, W is an odd wheel. Let x be the center of W, and $x_1 x_2 \cdots x_m$ be the circuit of $W - x$. Denote by X_i the triangle on $\{x, x_i, x_{i+1}\}$, $1 \leq i \leq m$, where $x_{m+1} = x_1$, and so W can be expressed as $X_1 X_2 \cdots X_m$. If $x_s x_t \in E(G)$ for some $s < t - 1$, then let T be the triangle on $\{x, x_s, x_t\}$. Since m is odd, either $T X_s X_{s+1} \cdots X_{t-1}$ or $T X_t X_{t+1} \cdots X_m X_1 \cdots X_{s-1}$ is an even wheel, which is, by Lemmas 2.6 and 2.4, a contradiction to the fact that G is not 3-flow contractible. This shows that W is an induced subgraph of G. If $G = W$, let G_1 be any edge of G, and then we have that $G = W \oplus_2 G_1$, as required. Suppose therefore that $|V(G)| > |V(W)|$.

Since G is triangularly connected, there is an edge $e^* \in E(W)$ contained in some triangle which has a vertex in $V(G) \setminus V(W)$. An edge $e \in E(G) \setminus E(W)$ is *reachable* from e^* if there is a triangle-path $H = T_1 T_2 \cdots T_m$ such that $e^* \in E(T_1)$, $e \in E(T_m)$, and $E(H) \cap E(W) = \{e^*\}$. Let R be the set of all the edges which is reachable from e^*. For simplicity, we also use R for the subgraph induced by R. Clearly, $V(e^*) \subseteq V(R)$.

If $V(R) \cap (V(W) \setminus V(e^*)) \neq \emptyset$, let $H = T_1 T_2 \cdots T_m$ be a triangle-path in $R \cup \{e^*\}$ such that $e^* \in E(T_1)$ and T_m contains a vertex in $V(R) \cap (V(W) \setminus V(e^*))$, and subject to this, m is as small as possible. Let $V(T_m) = \{a_1, a_2, a_3\}$, where $a_1 \in V(R) \cap (V(W) \setminus V(e^*))$. By the choice of m, $E(T_m) \cap E(T_{m-1}) = \{a_2 a_3\}$. Clearly, there is a triangle-path $Q = Q_1 Q_2 \cdots Q_t$ in W such that $e^* \in E(Q_1)$ and $a_1 \in V(Q_t)$. Then, $T_m T_{m-1} \cdots T_2 T_1 Q_1 \cdots Q_t$ is a triangle-path with the property described in Lemma 2.3, which together with Lemma 2.4 yields a contradiction. In what follows, we suppose that $V(R) \cap V(W) = V(e^*)$.

Since G is 3-connected, there must be an edge with exactly one end in $V(R) \setminus V(e^*)$, which is triangularly connected to edges in W. Let wa be such an edge with $a \in V(R) \setminus V(e^*)$ and $wa \notin R$. Since G is triangularly connected and W is an induced subgraph, there is a triangle-path $H = T_1 T_2 \cdots T_m$ such that $wa \in E(T_1)$, T_m contains some edge $e \in E(W)$ and $E(H) \cap E(W) = \{e\}$. Since $wa \notin R$, we see that $e \neq e^*$.

(i) $E(H) \cap R \neq \emptyset$. Let $e' \in E(H) \cap R$. If $e' \in E(T_i)$ for some $i < m$, then, since T_m is the only triangle in H which contains edges of W, we see that wa is reachable from e^* through a triangle-path from e^* to e' in R plus $T_i T_{i-1} \cdots T_1$, a contradiction. Thus, $e' \in E(T_m)$, which implies that the three edges e^*, e', e have a common end, denoted by z. Let P be a triangle-path in $R \cup \{e^*\}$ connecting e^* to e'. If $z = x$, then, since W is an odd wheel, e^* and e are connected by two triangle-paths W_1 and W_2 in W, whose lengths have different parity. Consequently, either $P \cup T_m \cup W_1$ or $P \cup T_m \cup W_2$ is a triangle-cycle of even length, which is, by Lemmas 2.6 and 2.4, a contradiction. Otherwise, $z = x_i$ for some i. If x is not the other end of e^*, say $e^* = x_i x_{i+1}$, then, $X_{i+1} X_{i+2} \cdots X_m X_1 \cdots X_{i-1} T_m P$ (recall that X_i is the triangle on $\{x, x_i, x_{i+1}\}$) is a triangle-path with the property as described in Lemma 2.3 (X_{i+1} and x_{i+1} play the same role as T_1 and x there), a contradiction. If x is the other end of e^*, then $e = x_i x_{i+1}$ or $x_i x_{i-1}$. Without loss of generality, suppose that $e = x_i x_{i+1}$. Then $X_{i+1} X_{i+2} \cdots X_m X_1 \cdots X_{i-1} P T_m$ is a triangle-path yielding a contradiction again.

(ii) $E(H) \cap R = \emptyset$. Let P be a triangle-path in $R \cup \{e^*\}$ from e^* to the vertex a, and let Q be a triangle-path connecting e^* and e in W. Note that $a \in V(T_1)$. Let t be the largest integer such that $a \in V(T_t)$. Then, $a \notin V(T_{t+1})$, and $T_t T_{t+1} \cdots T_m Q P$ is a triangle-path with the property as described in Lemma 2.3 (T_t and a play the same role as T_1 and x there), a contradiction again. This completes the proof of Theorem 1.4. ■

Proof of Theorem 1.5: If there are graphs W and G_1, as described in the theorem, such that $G = W \oplus_2 G_1$, then it follows from Lemma 2.1(i) that G does not have a nowhere-zero 3-flow. Conversely, Suppose that G has no nowhere-zero 3-flow. Thus, G is not 3-flow contractible, and by Theorem 1.4, there is an odd wheel W and a subgraph G_1 such that $G = W \oplus_2 G_1$, where G_1 is triangularly connected and not 3-flow contractible. If W is a triangle, let $E(W) = \{e_1, e_2, e_3\}$, where e_3 is the edge in $E(W) \cap E(G_1)$. Let $G^* = G/e_2$. Then G^* has a 2-circuit, which implies, by Lemma 2.4, that G^* is 3-flow contractible, and so has a nowhere-zero 3-flow f, which can be extended to a nowhere-zero 3-flow in G by assigning $f(e_1)$ to e_2. This is a contradiction, and thus, W is a nontrivial odd wheel. We claim that G_1 has no nowhere-zero 3-flow. If not so, let f_1 be a nowhere-zero 3-flow in G_1. It is easy to see that the removal of any edge from a nontrivial odd wheel results in a graph having a nowhere-zero 3-flow. Let f_2 be a nowhere-zero 3-flow in $W \setminus \{e_3\}$. Then the combination of f_1 and f_2 is a nowhere-zero 3-flow in G. This contradiction shows that G_1 has no nowhere-zero 3-flow, and completes the proof of Theorem 1.5. ■

4 Triangularly connected graphs

Theorem 4.1 *Let G be a triangularly connected graph with $|V(G)| \geq 3$. If G has no nowhere-zero 3-flow, then it contains at least four vertices of degree 3.*

Proof. We use induction on $|V(G)|$. Clearly, $|V(G)| \geq 4$. If $|V(G)| = 4$, then $G = K_4$, and the theorem holds. Suppose that $|V(G)| \geq 5$ and the theorem is true for any triangularly connected graph G' with $|V(G')| < |V(G)|$. If G is an odd wheel, then G has $|V(G)| - 1$ vertices of degree 3. Otherwise, by Theorem 1.5, there is an odd wheel W and a subgraph G_1 such that $G = W \oplus_2 G_1$, where G_1 is triangularly connected and has no nowhere-zero 3-flow. Since G is not an odd wheel, we have that $|V(G_1)| \geq 3$. By the induction hypothesis, G_1 has at least four vertices of degree 3. Clearly, W has at least four vertices of degree 3. But $|V(W) \cap V(G_1)| = 2$, and so G has at least four vertices of degree 3. ■

In the characterization of Theorem 1.4, we may have triangles, which give vertices of degree 2. In the proof of Theorem 4.1 above, if we use Theorem 1.4 instead of Theorem 1.5, then we have that

Theorem 4.2 *Let G be a triangularly connected graph with $|V(G)| \geq 3$. If G is not 3-flow contractible, then it contains either two vertices of degree 2 or at least three vertices of degree at most 3.*

Let G be the graph consisting of m copies $H_1, H_2, \cdots, H_m$ of K_4 such that $|V(H_i) \cap V(H_{i+1})| = 2$, $1 \leq i \leq m-1$, and $V(H_i) \cap V(H_j) = \emptyset$ if $j > i+1$. Then G is triangularly connected and has no nowhere-zero 3-flow. G has exactly 4 vertices of degree 3. In this sense, Theorem 4.1 is best possible. If H_1 and H_m are replaced by triangles, then we have an example showing Theorem 4.2 is best possible.

A connected graph G is *locally connected* if the neighbors of each vertex in G induce a connected subgraph. It is known that a locally connected graph is triangularly connected. Lai [9] proved that if the neighbors of each vertex in G induce a 3-edge-connected subgraph (locally 3-edge-connected), then G has a nowhere-zero 3-flow. This was extended by DeVos et al. [3], who prove that every triangularly connected graph with minimum degree at least 4 is 3-flow contractible [3, Theorem 1.4], which is now an immediate consequence of Theorem 4.2. Since the 3 edges incident with a vertex of degree 3 form an edge-cut, it is clear that Corollary 1.6 is an immediate consequence of Theorem 4.1.

5 Chordal graphs

A *chordal* graph is one that contains no induced circuit of length more than 3. Let G be a 2-edge-connected chordal graph. It is easy to see that if G is simple, then every edge of G is contained in a triangle. However, a 2-edge-connected chordal graph is not necessary to be triangularly connected. A *T-block* of a graph G is a maximal triangularly connected subgraph of G. Thus, if B is a T-block of G, then any subgraph with B as a proper subgraph is not triangularly connected. Clearly, if every edge of G is contained in a triangle, then $E(G)$ can be partitioned into edge-disjoint T-blocks. The following lemma shows that for a chordal graph, the T-blocks are identical with blocks (maximal nonseparable subgraphs).

Lemma 5.1 *Let G be a chordal graph. Two edges of G are contained in a T-block if and only if they lie on a common circuit.*

Proof. Let e and e' be two edges of G. If they are contained in a T-block B, then either $|V(B)| = 2$, in which e and e' lie on a 2-circuit, or $|V(B)| \geq 3$, in which B is 2-connected and e, e' lie on a common circuit. Conversely, suppose that e and e' lie on a common circuit C. We shall prove, by induction on $|E(C)|$, that e and e' are contained in a T-block. It is trivially true if $|E(C)| \leq 3$. Suppose therefore $|E(C)| \geq 4$ and the statement holds for any circuit C' with $|E(C')| < |E(C)|$. Since G is a chordal graph, C has a chord xy, where $x, y \in V(C)$ and $xy \notin E(C)$. Let P_1 and P_2 be the two segments of C divided by x and y. Then $C_i = P_i \cup \{xy\}$ is a circuit with $|E(C_i)| < |E(C)|$, $i = 1, 2$. If e and e' are contained in the same circuit C_i, then by the induction hypothesis, they are in the same T-block, and we are done. Without loss of generality, suppose thus that $e \in E(C_1)$ and $e' \in E(C_2)$. By the induction hypothesis, e and xy are contained in a T-block, and xy and e' are contained in a T-block, which implies that e and e' are in the same T-block. This completes the proof of the lemma. ■

Theorem 5.2 *A 2-edge-connected chordal graph has a nowhere-zero 3-flow if it contains at most three 3-cuts.*

Proof. Let G be a 2-edge-connected chordal graph without nowhere-zero 3-flow. We shall prove that G contains at least four 3-cuts. Let $B_1, B_2, \cdots, B_n$ be T-blocks of G. If each B_i has a nowhere-zero 3-flow f_i, $1 \leq i \leq n$, then the combination of f_i $(1 \leq i \leq n)$ is a nowhere-zero 3-flow in G. Thus, there is a T-block, say B_1, having no nowhere-zero 3-flow. Since G is 2-edge-connected, it follows from Lemma 5.1 that $|V(B_1)| \geq 3$. By Corollary 1.6, B_1 contains at least four 3-cuts, and by Lemma 5.1, each such cut is also an edge-cut of G, which gives four 3-cuts of G. This proves the theorem. ■

In the proof above, if the graph G is not 3-flow contractible, then at least one of the T-blocks is not 3-flow contractible. Using Theorem 4.2 instead of Corollary 1.6, the same arguments yield that

Theorem 5.3 *A 2-edge-connected chordal graph is 3-flow contractible if it contains at most one ℓ-cut for some $\ell \leq 3$.*

As an immediate consequence, Theorem 5.3 gives a result of Lai [8] that every 4-edge-connected chordal graph is 3-flow contractible.

6 Square of a graph

The *square* of a graph G, denoted by G^2, is the graph obtained from G by adding new edges joining every pair of vertices at distance 2 in G.

Proposition 6.1 *The square of a connected graph is triangularly connected.*

Proof. Let G be a connected graph. It is easy to see that the neighbors of a vertex of G induce a complete subgraph in G^2. Then G^2 is locally connected, and thus, it is triangularly connected by the connectedness of G. ∎

Let $\mathcal{F}$ be the set of nontrivial trees with maximum degree 3 and $\mathcal{F}^*$ the set of graphs which are trees in $\mathcal{F}$, or hamiltonian simple graphs on 4 vertices, or obtained from a tree T in $\mathcal{F}$ by adding edges between some leaves at distance 2 in T. The following theorem was proved by DeVos et al. [3], which strengthens a result of Xu and Zhang [11]. We present here an alternative proof, as an application of Theorem 1.4.

Theorem 6.2 *Let G be a connected graph. Then G^2 is not 3-flow contractible if and only if G is a member of $\mathcal{F}^*$.*

Proof. Suppose G is a member of $\mathcal{F}^*$. We shall prove, by induction on $|V(G)|$, that G^2 is not 3-flow contractible. If $|V(G)| \leq 4$, then G^2 is a single edge, or a triangle, or a K_4, or the 2-sum of two triangles, and so the theorem holds. Suppose now that $|V(G)| \geq 5$ and the theorem holds for every graph G' with $|V(G')| < |V(G)|$. Since $|V(G)| \geq 5$ and $G \in \mathcal{F}^*$, there is an edge e such that the deletion of e from G leaves two nontrivial components H_1 and H_2. Let $G_i = H_i \cup \{e\}$, $i = 1, 2$. Then $G_i \in \mathcal{F}^*$ and $|V(G_i)| < |V(G)|$, and by the induction hypothesis, G_i^2 is not 3-flow contractible, $i = 1, 2$. But $G^2 = G_1^2 \oplus_2 G_2^2$, and so G^2 is not 3-flow contractible by Lemma 2.1(ii).

Conversely, suppose that G^2 is not 3-flow contractible, we shall prove that $G \in \mathcal{F}^*$ by induction on $|V(G)|$. If $|V(G)| \leq 4$, it is clear that $G \in \mathcal{F}^*$. We assume thus that $|V(G)| \geq 5$ and the theorem holds for every G' with $|V(G')| < |V(G)|$. By Proposition 6.1, G^2 is triangularly connected. It follows from Theorem 1.4 that

$$G^2 = W \oplus_2 G_1,$$

where W is an odd wheel and G_1 is triangularly connected and not 3-flow contractible. If $|V(G_1)| = 2$, then $G^2 = W$. Let x be the center of W and $x_1x_2\cdots x_k$ the circuit of $W - x$. Since W is an odd wheel with $|V(W)| \geq 5$, we have that $k \geq 5$. By the definition of the square of a graph, the neighbors (in G) of x are consecutive on the circuit $x_1x_2\cdots x_k$, and thus $d_G(x) \leq 2$. Suppose that x_1 and x_2 are the possible neighbors of x in G. Now, the edge xx_4 implies that x_4 and x are at distance 2 in G, which is impossible. Thus, $|V(G_1)| \geq 3$. Suppose that $\{yz\} = E(W) \cap E(G_1)$. Note that $\{y, z\}$ forms a vertex-cut of G^2. By the definition of the square of a graph, it follows that $N_G(y)$ and $N_G(z)$ are entirely contained in different one of W and G_1, say that

$$N_G(y) \subseteq V(W) \text{ and } N_G(z) \subseteq V(G_1), \tag{6.1}$$

which implies that $yz \in E(G)$. Let H_1 and H_2 be the subgraphs of G such that $W = H_1^2$ and $G_1 = H_2^2$. By the induction hypothesis, $H_i \in \mathcal{F}^*$, $i = 1, 2$. Now, $G = H_1 \oplus_2 H_2$, which is, by (6.1), a member of $\mathcal{F}^*$. This completes the proof of Theorem 6.2. ∎

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