Fundamentals of Series: Table III: Basic Algebraic Techniques

From the seven unpublished manuscripts of H. W. Gould
Edited and Compiled by Jocelyn Quaintance

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1 Telescoping Series

1.1 Two Basic Telescoping Identities

Remark 1.1 *In this chapter, we assume, unless otherwise specified, that a is a nonnegative integer. We also assume that x and r are arbitrary real or complex numbers, unless otherwise specified.*

First Telescoping Identity
Let $u_k = f(k) - f(k+1)$. Then,

$$\boldsymbol{\sum_{k=a}^{n} u_k = f(a) - f(n+1), \qquad n \geq a} \tag{1.1}$$

Second Telescoping Identity
Let $v_k = f(k) + f(k+1)$. Then,

$$\boldsymbol{\sum_{k=a}^{n} (-1)^k v_k = (-1)^a f(a) - (-1)^{n+1} f(n+1), \qquad n \geq a} \tag{1.2}$$

1.1.1 Applications of Basic Telescoping Identities

$$\sum_{k=a}^{n-1} \frac{r^k}{(1+r^k x)(1+r^{k+1}x)} = \frac{1}{x(r-1)} \left(\frac{1}{1+r^a x} - \frac{1}{1+r^n x} \right), \tag{1.3}$$
$$n \geq a+1,\ r \neq 1,\ x \neq 0$$

$$\sum_{k=1}^{n-1} \frac{r^{k-1}}{(1-r^k)(1-r^{k+1})} = \frac{1}{r(1-r)} \left(\frac{1}{1-r} - \frac{1}{1-r^n} \right), \qquad r \neq 1 \tag{1.4}$$

$$\sum_{k=1}^{\infty} \frac{r^{k-1}}{(1-r^k)(1-r^{k+1})} = \begin{cases} \frac{1}{(1-r)^2}, & |r| < 1 \\ \frac{1}{r(1-r)^2}, & |r| > 1 \end{cases} \tag{1.5}$$

$$\sum_{k=a}^{n-1} \frac{1}{(r+kx)(r+(k+1)x)} = \frac{1}{x} \left(\frac{1}{r+ax} - \frac{1}{r+nx} \right), \qquad n \geq a+1,\ x \neq 0 \tag{1.6}$$

$$\sum_{k=1}^{n} \frac{1}{(r+kx)(r+(k+1)x)} = \frac{n}{(r+x)(r+(n+1)x)} \tag{1.7}$$

Hockey Stick Identity

$$\boldsymbol{\sum_{k=1}^{n} \binom{k-1}{m-1} = \binom{n}{m}}\text{, where } \boldsymbol{m} \text{ is a positive integer and } \boldsymbol{n \geq 1} \tag{1.8}$$

$$\sum_{k=a}^{n} (-1)^k \binom{x}{k} = (-1)^a \binom{x-1}{a-1} + (-1)^n \binom{x-1}{n}, \qquad n \geq a \tag{1.9}$$

$$\sum_{k=0}^{n} \binom{x-k}{n-k} = \binom{x+1}{n} \tag{1.10}$$

Remark 1.2 *In the following identity, we intepret $x!$ by the Gamma function, namely $\Gamma(x+1) = x!$, whenever x is not a negative integer.*

$$\sum_{j=a}^{n} \binom{j}{x} = \binom{n+1}{x+1} - \binom{a}{x+1}, \qquad n \geq a \tag{1.11}$$

$$\sum_{k=0}^{n} 2^{n-k} \binom{x+k}{k} \frac{x-k}{x+k} = \binom{x+n}{n}, \qquad x \neq 0 \tag{1.12}$$

Remark 1.3 *The following identity, proposed by M. S. Klamkin, is Problem 4561, p. 632 of the November 1953 (Vol. 60, No. 9) issue of Amer. Math. Monthly.*

$$\sum_{k=1}^{n} \frac{1}{k^2}\left(1-\frac{1}{\binom{k+r}{k}}\right) = \sum_{k=1}^{r} \frac{1}{k^2}\left(1-\frac{1}{\binom{k+n}{k}}\right), \qquad r \text{ and } n \text{ positive integers} \tag{1.13}$$

$$\sum_{k=1}^{n} \frac{1}{2k}\frac{2^{2k}}{\binom{2k}{k}} = \sum_{k=1}^{n} \frac{(-1)^k}{2k\binom{\frac{-1}{2}}{k}} = \frac{2^{2n}}{\binom{2n}{n}} - 1, \ \ n \geq 1 \tag{1.14}$$

$$\sum_{k=1}^{n} \binom{2k-2}{k}\frac{1}{k+1} = \frac{1}{3}\binom{2n}{n}\frac{1}{n+1} - \frac{1}{3}, \qquad n \geq 1 \tag{1.15}$$

$$\sum_{k=1}^{n} \frac{1}{(\sqrt{x+k}+\sqrt{x+k+1})\sqrt{(x+k)(x+k+1)}} = \frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x+n+1}} \tag{1.16}$$

$$\sum_{k=1}^{n} \frac{1}{(\sqrt{k}+\sqrt{k+1})\sqrt{k(k+1)}} = 1 - \frac{1}{\sqrt{n+1}}, \qquad n \geq 1 \tag{1.17}$$

$$\sum_{k=1}^{\infty} \frac{1}{(\sqrt{k}+\sqrt{k+1})\sqrt{k(k+1)}} = 1 \tag{1.18}$$

$$\sum_{k=0}^{\infty} \frac{1}{(\sqrt{x+k}+\sqrt{x+k+1})\sqrt{(x+k)(x+k+1)}} = \frac{1}{\sqrt{x}} \tag{1.19}$$

2 Summing Series by Recursive Formulas

2.1 Product Functions and Recursive Formulas

2.1.1 Product Function $f(n) = \prod_{k=1}^{n} \frac{1}{2k+1}$

Remark 2.1 *Note that $f(n) = \frac{1}{(2n+1)!!}$. For more information about double factorials, see the Wikipedia entry.*

$$\sum_{k=1}^{n} \frac{k}{\prod_{j=1}^{k}(2j+1)} = \sum_{k=1}^{n} \frac{kk!2^k}{(2k+1)!} = \frac{1}{2}\left(1 - \frac{1}{\prod_{k=1}^{n}(2k+1)}\right), \; n \geq 1 \tag{2.1}$$

$$\sum_{k=1}^{\infty} \frac{k}{\prod_{j=1}^{k}(2j+1)} = \sum_{k=1}^{\infty} \frac{kk!2^k}{(2k+1)!} = \frac{1}{2} \tag{2.2}$$

Remark 2.2 *In the following two identities, we evaluate any non-integer factorials by the Gamma function, i.e. $x! = \Gamma(x+1)$, whenever x is not a negative integer.*

$$\sum_{k=1}^{\infty} \frac{k}{2^k \left(k+\frac{1}{2}\right)!} = \frac{1}{\sqrt{\pi}} \tag{2.3}$$

$$\sum_{k=0}^{\infty} \frac{k}{2^k \left(k+\frac{1}{2}\right)!} = \sqrt{\pi} \tag{2.4}$$

$$\sum_{k=0}^{\infty} \frac{(k!)^2 2^k}{(2k+1)!} = \frac{\pi}{2} \tag{2.5}$$

2.1.2 Product Function $h(n) = \prod_{k=1}^{n} 2k$

Remark 2.3 *Note that $h(n) = (2n)!!$.*

$$\sum_{k=1}^{n} (2k-1)2^{k-1}(k-1)! = 2^n n! - 1, \qquad n \geq 1 \tag{2.6}$$

2.1.3 Product Function $g(n) = \prod_{k=1}^{n} \frac{1}{2k}$

Remark 2.4 *Note that* $g(n) = \frac{1}{(2n)!!}$.

$$\sum_{k=1}^{n} \frac{2k-1}{2^k k!} = 1 - \frac{1}{2^n n!}, \qquad n \geq 1 \tag{2.7}$$

$$\sum_{k=0}^{n} \frac{1-2k}{2^k k!} = \frac{1}{2^n n!} \tag{2.8}$$

$$\sum_{k=1}^{\infty} \frac{2k-1}{2^k k!} = 1 \tag{2.9}$$

2.1.4 Product Function $F(n) = \prod_{k=1}^{n} (2k+1)^p$

Remark 2.5 *For* $F(n)$*, we normally assume* p *is a nonnegative integer. However, we should note that the first identity in this subsection holds if* p *is any positive real number.*

$$\sum_{k=1}^{n} \left((2k+1)^p - 1\right) \frac{(2k-1)!^p}{(k-1)!^p 2^{pk-p}} = \frac{(2n+1)!^p}{n!^p 2^{pn}} - 1, \qquad n \geq 1 \tag{2.10}$$

$$\sum_{k=1}^{n} \frac{k(2k-1)!}{(k-1)!2^{k-1}} = \frac{(2n+1)!}{n!2^{n+1}} - \frac{1}{2}, \qquad n \geq 1 \tag{2.11}$$

$$\sum_{k=1}^{n} \binom{2k-1}{k} \frac{k^2(k+1)}{2^{2k}} (2k-1)! = \frac{(2n+1)!^2}{n!^2 2^{2n}} - 1, \qquad n \geq 1 \tag{2.12}$$

$$\sum_{k=1}^{n} \sum_{j=1}^{p} \binom{p}{j} (2k)^j \left(\frac{(2k-1)!}{(k-1)!2^{k-1}}\right)^p = \left(\frac{(2n+1)!}{n!2^n}\right)^p - 1 \tag{2.13}$$

2.1.5 Product Function $G(n) = \prod_{k=1}^{n}(2k+1)^{-p}$

Remark 2.6 *For $G(n)$, we assume p is a nonnegative integer.*

$$\sum_{k=1}^{n}\sum_{j=1}^{p}\binom{p}{j}(2k)^j\left(\frac{k!2^k}{(2k+1)!}\right)^p = 1 - \left(\frac{n!2^n}{(2n+1)!}\right)^p \tag{2.14}$$

$$\sum_{k=1}^{\infty}\sum_{j=1}^{p}\binom{p}{j}(2k)^j\left(\frac{k!2^k}{(2k+1)!}\right)^p = 1 \tag{2.15}$$

2.2 Evaluation of $\sum_{k=0}^{n}\binom{n}{k}^q$

Remark 2.7 *The formulas in this section can be found in Tor. B. Staver's "Om summasjon av potenser av binomialkoeffisienten", Norsk Matematisk Tidsskrift, 29.Årgang, 1947, pp. 97-103.*

2.2.1 Let $q = -1$

$$\sum_{k=0}^{n}\frac{k}{\binom{n}{k}} = \frac{n}{2}\sum_{k=0}^{n}\frac{1}{\binom{n}{k}} \tag{2.16}$$

$$\sum_{k=0}^{n}\frac{1}{\binom{n}{k}} = \frac{n+1}{2n}\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}} + 1 \tag{2.17}$$

$$\sum_{k=0}^{n}\frac{1}{\binom{n}{k}} = \frac{n+1}{2^{n+1}}\sum_{j=1}^{n+1}\frac{2^j}{j} \tag{2.18}$$

$$\sum_{k=0}^{n}\frac{1}{\binom{n}{k}} = \frac{(n+1)}{S_{n+1}(1)}\sum_{k=1}^{n+1}\frac{S_k(1)}{k}, \text{ where } S_n(1) = \sum_{k=0}^{n}\binom{n}{k} = 2^n \tag{2.19}$$

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{\binom{n}{k}} = 2 \tag{2.20}$$

$$\lim_{n\to\infty}\frac{n}{2^n}\sum_{k=1}^{n}\frac{2^k}{k} = 2 \tag{2.21}$$

2.2.2 Let $q = -2$

$$\sum_{k=0}^{n} \frac{1}{\binom{n}{k}^2} = \frac{(n+2)^3}{2(2n+5)(n+1)^2} \sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}^2} + \frac{3(n+2)}{2n+5} \tag{2.22}$$

$$\sum_{k=0}^{n} \frac{1}{\binom{n}{k}^2} = \frac{3(n+1)^2}{2n+3} \frac{1}{\binom{2n+2}{n+1}} \sum_{k=1}^{n+1} \binom{2k}{k} \frac{1}{k} \tag{2.23}$$

$$\sum_{k=0}^{n} \frac{1}{\binom{n}{k}^2} = \frac{3(n+1)^2}{2n+3} \frac{1}{S_{n+1}(2)} \sum_{k=1}^{n+1} \frac{S_k(2)}{k}, \text{ where } S_n(2) = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \tag{2.24}$$

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{\binom{n}{k}^2} = 2 \tag{2.25}$$

$$\lim_{n\to\infty} \frac{3(n+1)^2}{2n+3} \frac{1}{\binom{2n+2}{n+1}} \sum_{k=1}^{n+1} \binom{2k}{k} \frac{1}{k} = 2 \tag{2.26}$$

2.2.3 Evaluation of $\sum_{k=0}^{n} \frac{x^k}{\binom{n}{k}}$

$$\sum_{k=0}^{n} \frac{x^k}{\binom{n}{k}} = (n+1) \left(\frac{x}{1+x}\right)^{n+1} \sum_{k=1}^{n+1} \frac{1+x^k}{k(1+x)} \left(\frac{1+x}{x}\right)^k, \qquad x \neq -1 \tag{2.27}$$

$$\left(1+\frac{1}{x}\right) \sum_{k=0}^{n+1} \frac{x^k}{\binom{n+1}{k}} = \frac{n+2}{n+1} \sum_{k=0}^{n} \frac{x^k}{\binom{n}{k}} + x^{n+1} + \frac{1}{x}, \qquad x \neq 0 \tag{2.28}$$

3 Rational Functions and Partial Fraction Decompositions

Remark 3.1 *In this chapter, we assume, unless otherwise specified, that* r *is a positive integer.*

3.1 Evaluation of $\sum_{k=1}^{n} \frac{1}{k(k+1)\dots(k+r)}$

$$\sum_{k=1}^{n} \frac{1}{\prod_{i=0}^{r}(k+i)} = \frac{1}{r}\left(\frac{1}{r!} - \frac{1}{\prod_{j=1}^{r}(n+j)}\right), \qquad n, r \geq 1 \tag{3.1}$$

3.1.1 Applications of Equation (3.1)

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}, \qquad n \geq 1 \tag{3.2}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right), \qquad n \geq 1 \tag{3.3}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)(k+3)} = \frac{1}{3}\left(\frac{1}{6} - \frac{1}{(n+1)(n+2)(n+3)}\right), \qquad n \geq 1 \tag{3.4}$$

$$\sum_{k=1}^{n} \frac{1}{\prod_{i=0}^{r}(k+i)} = \sum_{k=1}^{n}\sum_{t=0}^{r} \frac{(-1)^t}{t!(r-t)!(k+t)}, \qquad n, r \geq 1 \tag{3.5}$$

$$\sum_{k=1}^{n}\sum_{t=0}^{r}(-1)^t\binom{r}{t}\frac{1}{k+t} = \frac{1}{r} - n\sum_{t=0}^{r}(-1)^t\binom{r}{t}\frac{1}{n+t}, \qquad n, r \geq 1 \tag{3.6}$$

3.1.2 Connections to the Gamma Function

Remark 3.2 *Let* p *and* q *be any real or complex numbers which are not negative integers. Then,*

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \frac{(p-1)!(q-1)!}{(p+q-1)!}, \tag{3.7}$$

is an expression for the Beta Function.

$$\beta(n+1,m+1) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1}{m+k+1} \tag{3.8}$$

$$\sum_{k=1}^{n} \beta(r+1,k) = \frac{1}{r} - n\beta(r+1,n), \qquad n \geq 1 \tag{3.9}$$

3.2 Evaluation of $\sum_{k=1}^{n} \frac{1}{(k+a)(k+a+1)\ldots(k+a+r)}$

Remark 3.3 *In Section 3.2, we assume a is any nonzero real or complex number which is not a negative integer. Note that $a! = \Gamma(a+1)$.*

$$\boldsymbol{\sum_{k=1}^{n} \prod_{i=0}^{r} \frac{1}{k+a+i} = \sum_{k=1}^{n} \frac{(k+a-1)!}{(k+a+r)!}} \tag{3.10}$$
$$\boldsymbol{= \frac{1}{r}\left(\frac{a!}{(a+r)!} - \frac{(n+a)!}{(n+a+r)!}\right),\ n, r \geq 1}$$

3.2.1 Applications of Equation (3.10)

$$\sum_{k=1}^{n} \frac{1}{(k+a)(k+a+1)} = \frac{n}{(a+1)(n+a+1)}, \qquad n \geq 1 \tag{3.11}$$

$$\sum_{k=1}^{n} \frac{1}{(k+a)(k+a+1)(k+a+2)} = \frac{1}{2}\left(\frac{1}{(a+1)(a+2)} - \frac{1}{(n+a+1)(n+a+2)}\right) \tag{3.12}$$

$$\sum_{k=0}^{n} \binom{k+a}{r} = \binom{n+a+1}{r+1} - \binom{a}{r+1} \tag{3.13}$$

$$\sum_{k=0}^{n} \binom{k-a}{r} = \binom{n-a+1}{r+1} + (-1)^r \binom{a+r}{r+1} \tag{3.14}$$

$$\sum_{k=0}^{n} \frac{(k+n)!}{k!} = \frac{(2n+1)!}{(n+1)!} \tag{3.15}$$

$$\sum_{k=0}^{n} \frac{(k+a)!}{k!} = \frac{(n+a+1)!}{(a+1)n!} \tag{3.16}$$

$$\sum_{k=0}^{n} (k+1) = \frac{(n+2)(n+1)}{2} \tag{3.17}$$

$$\sum_{k=0}^{n} (k+1)(k+2) = \frac{(n+3)(n+2)(n+1)}{3} \tag{3.18}$$

$$\sum_{k=0}^{n} \binom{k+n}{k} = \binom{2n+1}{n+1} \tag{3.19}$$

$$\sum_{k=0}^{n} \binom{k+a}{k} = \binom{n+a+1}{n} \tag{3.20}$$

$$\sum_{k=0}^{n} \binom{3n-k}{2n} = \binom{3n+1}{n} \tag{3.21}$$

3.2.2 Connections to Pascal's Formulas

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad n \geq 1 \tag{3.22}$$

$$\sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6}, \qquad n \geq 1 \tag{3.23}$$

$$\sum_{k=1}^{n} \frac{k(k+1)(k+2)}{6} = \frac{n(n+1)(n+2)(n+3)}{24}, \qquad n \geq 1 \tag{3.24}$$

Pascal's Formula

$$\boldsymbol{\sum_{k=1}^{n} \binom{k+r-1}{k-1} = \binom{n+r}{r+1},} \qquad \text{where } \boldsymbol{r} \text{ is a nonnegative integer} \tag{3.25}$$

3.2.3 Reciprocal Applications of Equation (3.10)

$$\sum_{k=1}^{n} \frac{1}{\binom{k+a+r}{r+1}} = \left(1+\frac{1}{r}\right)\left(\frac{1}{\binom{a+r}{r}} - \frac{1}{\binom{n+a+r}{r}}\right), \qquad n \geq 1 \tag{3.26}$$

$$\sum_{k=0}^{n} \frac{1}{\binom{k+a+r}{r+1}} = \left(1+\frac{1}{r}\right)\left(\frac{1}{\binom{a+r-1}{r}} - \frac{1}{\binom{n+a+r}{r}}\right) \tag{3.27}$$

$$\sum_{k=1}^{\infty} \frac{1}{\binom{k+a+r}{r+1}} = \left(1+\frac{1}{r}\right)\frac{1}{\binom{a+r}{r}} \tag{3.28}$$

$$\sum_{k=1}^{\infty} \frac{1}{\binom{k+r+1}{r+1}} = \frac{1}{r} \tag{3.29}$$

$$\sum_{k=1}^{\infty} \frac{1}{\binom{2n+k}{n+1}} = \left(1+\frac{1}{n}\right)\frac{1}{\binom{2n}{n}} = \frac{1}{\binom{2n}{n-1}}, \qquad n \geq 1 \tag{3.30}$$

$$\sum_{k=1}^{n} \frac{1}{\binom{2n+k}{n}} = \frac{n}{n-1}\left(\frac{1}{\binom{2n}{n-1}} - \frac{1}{\binom{3n}{n-1}}\right), \qquad n \geq 1 \tag{3.31}$$

$$\sum_{k=1}^{n} \frac{1}{\binom{k+r}{k}} = \left(1+\frac{1}{r-1}\right)\left(\frac{1}{r} - \frac{1}{\binom{n+r}{r-1}}\right), \qquad n \geq 1 \tag{3.32}$$

$$\sum_{k=0}^{n} \frac{1}{\binom{k+r}{k}} = \left(1+\frac{1}{r-1}\right)\left(1 - \frac{1}{\binom{n+r}{r-1}}\right) \tag{3.33}$$

$$\sum_{k=1}^{\infty} \frac{1}{\binom{k+r}{k}} = \sum_{k=1}^{\infty} \frac{1}{\prod_{j=1}^{r}\left(1+\frac{k}{j}\right)} = \frac{1}{r-1} \tag{3.34}$$

$$\sum_{k=1}^{\infty} \frac{1}{\binom{k+r}{k} k} = \frac{1}{r} \tag{3.35}$$

$$\sum_{k=0}^{\infty} \frac{1}{\binom{a+k}{k}} = \frac{a}{a-1}, \qquad a \neq 1 \tag{3.36}$$

Finite Harmonic Series Formula

$$\sum_{j=1}^{n} \frac{1}{j} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{j=1}^{n} \frac{1}{\binom{j+k}{k}}, \qquad n \geq 1 \tag{3.37}$$

3.3 Evaluation of $\sum_{k=a}^{n} \frac{1}{(k-x)(k-x+1)}$

Remark 3.4 *In Section 3.3, we assume a is any nonnegative integer and x is a nonzero real or complex number for which the series are defined.*

$$\boldsymbol{\sum_{k=a}^{n} \frac{1}{(k-x)(k-x+1)} = \frac{1}{a-x} - \frac{1}{n-x+1}} \tag{3.38}$$

3.3.1 Derivatives of Equation (3.38)

$$\sum_{k=a}^{n} \frac{2k-2x+1}{(k-x)^2(k-x+1)^2} = \frac{1}{(a-x)^2} - \frac{1}{(n-x+1)^2} \tag{3.39}$$

$$\sum_{k=a}^{n} \frac{2k+1}{k^2(k+1)^2} = \frac{1}{a^2} - \frac{1}{(n+1)^2}, \qquad a \neq 0 \tag{3.40}$$

$$\sum_{k=a}^{n} \frac{2k+3}{(k+1)^2(k+2)^2} = \frac{1}{(a+1)^2} - \frac{1}{(n+2)^2} \tag{3.41}$$

3.4 Evaluation of $\sum_{k=1}^{n} \frac{1}{k(k+r)}$

$$\boldsymbol{\sum_{k=1}^{n} \frac{1}{k(k+r)} = \frac{1}{r} \sum_{i=1}^{r} \frac{n}{i(i+n)}, \qquad n \geq 1} \tag{3.42}$$

3.4.1 Applications of Equation (3.42)

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}, \qquad n \geq 1 \tag{3.43}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right), \qquad n \geq 1 \tag{3.44}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+3)} = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right), \qquad n \geq 1 \tag{3.45}$$

Harmonic Series Formula

$$\sum_{r=1}^{n} \frac{1}{r} = n \sum_{k=1}^{\infty} \frac{1}{k(k+n)}, \qquad n \geq 1 \tag{3.46}$$

Generalization of Equation (3.42)

$$\sum_{k=a}^{n} \frac{1}{k(k+r)} = \frac{1}{r} \sum_{i=1}^{r} \frac{n-a+1}{(a+i-1)(n+i)}, \qquad \text{where } a \text{ is a positive integer} \tag{3.47}$$

3.5 Evaluation of $\sum_{k=1}^{n} \frac{r}{(b+k)(b+k+r)}$

Remark 3.5 *In Section 3.5, we assume b is a positive integer.*

$$\boldsymbol{\sum_{k=1}^{n} \frac{r}{(b+k)(b+k+r)} = \sum_{k=1}^{r} \frac{n}{(b+k)(b+k+n)}, \qquad n \geq 1} \tag{3.48}$$

3.5.1 Applications of Equation (3.48)

$$\sum_{k=0}^{n-1} \frac{1}{(b+k)(b+k+1)} = \frac{n}{b(n+b)}, \qquad n \geq 1 \tag{3.49}$$

Remark 3.6 *In the following identity, we assume p and q are positive integers.*

$$\sum_{k=0}^{n-1} \frac{1}{(p+qk)(p+q(k+1))} = \frac{n}{p(p+qn)}, \qquad n \geq 1 \tag{3.50}$$

3.6 Evaluation of $\sum \frac{1}{n^2-a^2}$

Remark 3.7 *In Section 3.6, we assume a is a positive integer.*

$$\sum_{n=a+1}^{\infty} \frac{1}{n^2-a^2} = \frac{1}{2a}\sum_{k=1}^{2a}\frac{1}{k} \tag{3.51}$$

$$\sum_{\substack{k=1\\k\neq n}}^{\infty} \frac{1}{n^2-k^2} = \frac{-3}{4n^2}, \qquad n \geq 1 \tag{3.52}$$

4 Pascal's Identity in Evaluation of Series

Remark 4.1 *In this chapter, we assume, unless otherwise specified, that x is an arbitrary real or complex number. We will also assume k is a nonnegative integer, while r is a positive integer.*

4.1 Iterations of Pascal's Identity

4.1.1 Calculations Involving $\sum_{t=0}^{r-1}(-1)^t\binom{x+1}{k-t}$

$$\boldsymbol{\sum_{t=0}^{r-1}(-1)^t\binom{x+1}{k-t} = \binom{x}{k} - (-1)^r\binom{x}{k-r}} \tag{4.1}$$

$$\sum_{t=0}^{\infty}(-1)^t\binom{x+1}{k-t} = \binom{x}{k} \tag{4.2}$$

$$\sum_{t=0}^{n-1}(-1)^t\binom{n+1}{n-t} = 1-(-1)^n \tag{4.3}$$

4.1.2 Calculations Involving $\sum_{t=1}^{r}(-1)^{t-1}\binom{x+1}{k+t}$

$$\boldsymbol{\sum_{t=1}^{r}(-1)^{t-1}\binom{x+1}{k+t} = \binom{x}{k} - (-1)^r\binom{x}{k+r}} \tag{4.4}$$

$$\sum_{t=0}^{r}(-1)^t\binom{x+1}{k+t} = \binom{x}{k-1} + (-1)^r\binom{x}{k+r} \tag{4.5}$$

$$\sum_{t=0}^{r}(-1)^t\binom{x+1}{t}=(-1)^r\binom{x}{r} \tag{4.6}$$

4.1.3 Calculations Involving $\sum_{t=0}^{r-1}\binom{x+t}{k-1}$

$$\boldsymbol{\sum_{t=0}^{r-1}\binom{x+t}{k-1}=\binom{x+r}{k}-\binom{x}{k}, \qquad k\geq 1} \tag{4.7}$$

$$\sum_{t=0}^{n-1}\binom{n+t}{k-1}=\binom{2n}{k}-\binom{n}{k} \tag{4.8}$$

Remark 4.2 *The following identity is found in "The class of the free metabelian group with exponent p^2", by S. Bachmuth and H. Y. Mochizuki, Communications on Pure and Applied Math., Vol. 21 (1968), pp. 385-399.*

$$\sum_{k=0}^{n}\binom{k}{i}\binom{k}{j}=\sum_{k=0}^{i}\binom{i}{k}\binom{j}{k}\binom{n+k+1}{i+j+1}, \qquad 0\leq i\leq j \tag{4.9}$$

4.1.4 Calculations Involving $\sum_{k=1}^{n}\binom{2x}{2k}$

Remark 4.3 *In this subsection, we assume p is a nonnegative integer. We also assume $[x]$ is the greatest integer in x.*

$$\boldsymbol{\sum_{k=1}^{p}\binom{2x}{2k}=\sum_{k=1}^{2p}\binom{2x-1}{k}} \tag{4.10}$$

$$\boldsymbol{\sum_{k=0}^{n}\binom{2n}{2k}=\sum_{k=0}^{2n}\binom{2n-1}{k}=\begin{cases}2^{2n-1}, & n\neq 0\\ 1, & n=0\end{cases}} \tag{4.11}$$

$$\sum_{k=0}^{n}\binom{2n}{2k}=\sum_{k=0}^{n}\binom{2n}{2k+1}=2^{2n-1}, \qquad n\neq 0 \tag{4.12}$$

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}=\sum_{k=0}^{[\frac{n-1}{2}]}\binom{n}{2k+1}=2^{n-1}, \qquad n\geq 1 \tag{4.13}$$

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} = 2^{2n} \tag{4.14}$$

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} k = (2n-1)2^{2n-2}, \qquad n \geq 1 \tag{4.15}$$

$$\sum_{k=0}^{n} \binom{2n+1}{2k} = \sum_{k=0}^{n} \binom{2n+1}{2k+1} = 2^{2n} \tag{4.16}$$

4.2 Generalizations of Equation (4.1)

$$\boldsymbol{\sum_{j=0}^{r} (-1)^j \binom{x}{r-j} = \binom{x-1}{r}, \qquad r \geq 0} \tag{4.17}$$

$$\sum_{j=0}^{r} (-1)^j \binom{x}{j} = (-1)^r \binom{x-1}{r}, \qquad r \geq 0 \tag{4.18}$$

$$\boldsymbol{\sum_{k=0}^{n} (-1)^k \binom{x}{k} = \prod_{k=1}^{n} \left(1 - \frac{x}{k}\right) = (-1)^n \binom{x-1}{n} = \binom{n-x}{n}} \tag{4.19}$$

4.2.1 Applications of Equation (4.19)

$$\sum_{k=1}^{n} (-1)^k \binom{x}{k} k = (-1)^n x \binom{x-2}{n-1}, \qquad n \geq 1 \tag{4.20}$$

With the $\frac{1}{2}$ Transformation

$$\sum_{k=0}^{n} (-1)^k \binom{\frac{1}{2}}{k} = (-1)^n \binom{-\frac{1}{2}}{n} \tag{4.21}$$

Remark 4.4 *In the following identity, we evaluate any non-integral factorial value via the Gamma function, i.e.* $\Gamma(x) = (x-1)!$ *whenever* x *is not a negative integer.*

$$\sum_{k=0}^{n} (-1)^k \frac{1}{k!\left(\frac{1}{2}-k\right)!} = \frac{2(-1)^n}{n!\left(-\frac{1}{2}-n\right)!} \tag{4.22}$$

$$\sum_{k=0}^{n} \binom{2k}{k} \frac{1}{2^{2k}(2k-1)} = \frac{-1}{2^{2n}} \binom{2n}{n} \tag{4.23}$$

With the $\frac{-1}{2}$ Transformation

$$\sum_{k=0}^{n} \binom{2k}{k} \frac{1}{2^{2k}} = \frac{2n+1}{2^{2n}} \binom{2n}{n} \tag{4.24}$$

Remark 4.5 *The following two identities are the solution to Problem E995, p. 700 of the December 1951 American Mathematical Monthly.*

$$\sum_{k=1}^{n} \binom{2k}{k} \frac{k}{2^{2k}} = \frac{n(2n+1)}{3 * 2^{2n}} \binom{2n}{n}, \qquad n \geq 1 \tag{4.25}$$

$$\sum_{k=1}^{n} \binom{2k}{k} \frac{k}{2^{2k}} = \frac{n(n+1)}{3 * 2^{2n+1}} \binom{2n+2}{n+1}, \qquad n \geq 1 \tag{4.26}$$

$$\sum_{k=1}^{n} \binom{2k}{k} \frac{k^2}{2^{2k}} = \frac{n(2n+1)(3n+2)}{3 * 5 * 2^{2n}} \binom{2n}{n}, \qquad n \geq 1 \tag{4.27}$$

$$\sum_{k=1}^{n} \binom{2k}{k} \frac{k^3}{2^{2k}} = \frac{n(2n+1)(15n^2+18n+2)}{3 * 5 * 7 * 2^{2n}} \binom{2n}{n}, \qquad n \geq 1 \tag{4.28}$$

Two Identities using the Vandermonde Convolution

Remark 4.6 *In the following two identities, we assume α is a positive integer.*

$$\sum_{\alpha=1}^{n} \binom{n+1}{\alpha} \binom{n-1}{\alpha-1} = \binom{2n}{n}, \qquad n \geq 1 \tag{4.29}$$

$$\sum_{\alpha=1}^{n} \binom{n}{\alpha}^2 \frac{\alpha}{n-\alpha+1} = \binom{2n}{n+1}, \qquad n \geq 1 \tag{4.30}$$

Sparre-Andersen Formulas

Remark 4.7 *Let $f(x)$ be a given function and $F(k) = \sum_{j=0}^{k}(-1)^j \binom{k}{j} f(j)$. Then,*

$$\sum_{k=0}^{n}(-1)^k \binom{x}{k} F(k) = (-1)^n \binom{x-1}{n} \sum_{j=0}^{n}(-1)^j \binom{n}{j} \frac{x}{x-j} f(j), \tag{4.31}$$

where x is an arbitrary real or complex number such that $x - j \neq 0$ for $0 \leq j \leq n$.

Remark 4.8 *The following two identities are found in Erik Sparre-Andersen's "Two Summation Formulae for Product Sums of Binomial Coefficients", Mathematica Scandinavica, Vol. 1, 1953, pp. 261-262.*

$$\begin{aligned}\sum_{k=0}^{\alpha} \binom{x}{k}\binom{-x}{n-k} &= \frac{\alpha - x}{n}\binom{x}{\alpha}\binom{-x-1}{n-\alpha-1} \\ &= \frac{n-\alpha}{n}\binom{x-1}{\alpha}\binom{-x}{n-\alpha} = \frac{-\alpha-1}{n}\binom{x}{\alpha+1}\binom{-x-1}{n-\alpha-1},\end{aligned} \tag{4.32}$$

for all integers n, α such that $n \geq 1$ and $0 \leq \alpha \leq n$.

$$\sum_{k=0}^{\alpha} \binom{x}{k}\binom{1-x}{n-k} = \frac{(n-1)(1-x) - \alpha}{n(n-1)}\binom{x-1}{\alpha}\binom{-x}{n-\alpha-1}, \tag{4.33}$$

for all integers n, α such that $n \geq 2$ and $0 \leq alpha \leq n-1$.

Remark 4.9 *In the following two identities, assume x and z are arbitrary real or complex numbers, where $x - j \neq 0$ whenever $0 \leq j \leq \alpha$.*

$$\sum_{k=0}^{\alpha} \binom{x}{k}\binom{z}{n-k} = (-1)^\alpha \binom{x-1}{\alpha} \sum_{j=0}^{\alpha}(-1)^j \binom{\alpha}{j}\binom{z+j}{n}\frac{x}{x-j} \tag{4.34}$$

$$\sum_{k=0}^{\alpha} \binom{-x}{k}\binom{z}{n-k} = \binom{x+\alpha}{\alpha} \sum_{j=0}^{\alpha}(-1)^j \binom{\alpha}{j}\binom{z+j}{n}\frac{x}{x+j} \tag{4.35}$$

$$\begin{aligned}\sum_{k=\alpha+1}^{n} \binom{x}{k}\binom{-x}{n-k} &= \sum_{k=0}^{\alpha} \binom{x}{k}\binom{-x}{n-k} \\ &= \frac{x-\alpha}{n}\binom{x}{\alpha}\binom{-x-1}{n-\alpha-1}, \qquad n \geq 1, 0 \leq \alpha < n\end{aligned} \tag{4.36}$$

Saalschütz Formula

$$\sum_{k=a}^{n}(-1)^{k-a}\binom{n}{k}\frac{1}{k} = \sum_{k=a}^{n}\binom{k-1}{a-1}\frac{1}{k}, \qquad 1 \leq a \leq n \tag{4.37}$$

Derivatives Involving Identity (4.19)

Remark 4.10 *For the following three identities, we refer the reader to Version 1 of the Generalized Chain Rule provided in Section 3.6. Note that we let $D_x^n f(x)$ denote the n^{th} derivative of $f(x)$ with respect to x. Furthermore, we assume x, and z are real or complex numbers, while r is a real number.*

$$D_x^n z^r = \sum_{k=0}^{n} (-1)^{n-k} \binom{r}{k} \binom{r-k-1}{n-k} z^{r-k} D_x^n z^k \tag{4.38}$$

$$D_x^n z^{-r} = \sum_{k=0}^{n} (-1)^{k} \binom{r+k-1}{k} \binom{r+n}{n-k} \frac{1}{z^{r+k}} D_x^n z^k \tag{4.39}$$

$$D_x^n z^{-r} = r\binom{r+n}{n} \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \frac{1}{r+k} \frac{1}{z^{r+k}} D_x^n z^k \tag{4.40}$$

5 Reciprocal Pascal's Identity with Series

5.1 Additive Forms of Reciprocal Pascal's Identity

$$\boldsymbol{\frac{1}{\binom{n+1}{k}} + \frac{1}{\binom{n+1}{k+1}} = \frac{n+2}{n+1} \frac{1}{\binom{n}{k}}} \tag{5.1}$$

$$\frac{1}{\binom{2n+1}{k}} + \frac{1}{\binom{2n+1}{k+1}} = \frac{2n+2}{2n+1} \frac{1}{\binom{2n}{k}} \tag{5.2}$$

5.1.1 Applications of Equation (5.2)

$$\sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{\binom{2n}{k}} = \frac{1}{n+1}, \qquad n \geq 1 \tag{5.3}$$

$$\sum_{k=1}^{2n-1} \frac{(-1)^{k-1} k}{\binom{2n}{k}} = \frac{n}{n+1}, \qquad n \geq 1 \tag{5.4}$$

$$\sum_{k=1}^{2n-1} \frac{(-1)^{k-1} (n-k)}{\binom{2n}{k}} = 0, \qquad n \geq 1 \tag{5.5}$$

$$\sum_{k=1}^{2n-1} \frac{(-1)^{k-1}(n+k)}{\binom{2n}{k}} = \frac{2n}{n+1}, \qquad n \geq 1 \tag{5.6}$$

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{\binom{2n}{k}} = \frac{1}{2(n+1)} + \frac{(-1)^{n-1}}{2\binom{2n}{n}}, \qquad n \geq 1 \tag{5.7}$$

5.2 Subtraction Form of Reciprocal Pascal's Identity

$$\boldsymbol{\frac{1}{\binom{n+1}{k}} - \frac{1}{\binom{n+1}{k+1}} = \frac{n-2k}{n+1}\frac{1}{\binom{n}{k}}} \tag{5.8}$$

5.2.1 Applications of Equation (5.8)

$$\sum_{k=1}^{r} \frac{n-2k}{\binom{n}{k}} = 1 - \frac{n+1}{\binom{n+1}{r+1}}, \qquad n, r \geq 1 \tag{5.9}$$

$$\sum_{k=1}^{n} \frac{2k-n}{\binom{n}{k}} = n, \qquad n \geq 1 \tag{5.10}$$

$$\sum_{k=1}^{n-1} \frac{2k-n}{\binom{n}{k}} = 0, \qquad n \geq 2 \tag{5.11}$$

$$\sum_{k=0}^{n} \frac{n-2k}{\binom{n}{k}} = 0 \tag{5.12}$$

$$\sum_{k=0}^{n} \frac{(-1)^k}{\binom{n}{k}} = (1+(-1)^n)\,\frac{n+1}{n+2} \tag{5.13}$$

5.3 Generalized Additive Form of Reciprocal Pascal's Identity

$$\boldsymbol{\frac{1}{\binom{x}{k}} = \frac{x+1}{x+2}\left(\frac{1}{\binom{x+1}{k}} + \frac{1}{\binom{x+1}{k+1}}\right), \qquad x \neq -1, -2} \tag{5.14}$$

5.3.1 Applications of Equation (5.14)

$$\sum_{k=0}^{n} \frac{(-1)^k}{\binom{x}{k}} = \frac{x+1}{x+2}\left(1 + \frac{(-1)^n}{\binom{x+1}{n+1}}\right), \qquad x \neq -1, -2 \tag{5.15}$$

$$\sum_{k=0}^{n} \frac{2^{2k}}{\binom{2k}{k}} = \frac{1}{3}\left(1 + \frac{(2n+1)2^{2n+2}}{\binom{2n+2}{n+1}}\right) \tag{5.16}$$

$$\sum_{k=0}^{n} \frac{1}{\binom{x+k}{k}} = \frac{x}{x-1}\left(1 - \frac{1}{\binom{x+n}{n+1}}\right), \qquad x \neq 0, 1 \tag{5.17}$$

$$\sum_{k=j}^{n} \frac{1}{\binom{k}{j}} = \begin{cases} \sum_{k=1}^{n} \frac{1}{k}, & j = 1 \\ \frac{j}{j-1}\left(1 - \frac{1}{\binom{n}{j-1}}\right), & j > 1 \end{cases} \tag{5.18}$$

$$\sum_{k=j}^{\infty} \frac{1}{\binom{k}{r}} = \frac{j}{(r-1)\binom{j}{r}}, \qquad r > 1 \tag{5.19}$$