# Fundamentals of Series: Table II: Examples of Series Which Appear in Calculus

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## 1 The Binomial Theorem

**Remark 1.1** *In this table, unless otherwise specified, $n$ and $r$ are nonnegative intergers, and $x$ and $z$ are arbitrary real or complex numbers. We also assume that for any real number $x$, $[x]$ is the greatest integer in $x$.*

### 1.1 Binomial Theorem

#### 1.1.1 Basic Form with Integer Power

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} a^k, \text{ where } a \text{ is an arbitrary real or complex number} \tag{1.1}$$

#### 1.1.2 Newton's Binomial Theorem

$$(1+x)^z = \sum_{k=0}^{\infty} \binom{z}{k} x^k, \text{ where } z \text{ is a real or complex number and } |x| < 1 \tag{1.2}$$

#### 1.1.3 Applications of Newton's Binomial Theorem

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{x^k}{2^{2k}} = \frac{1}{\sqrt{1-x}}, \qquad |x| < 1 \tag{1.3}$$

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{2^{3k}} = \sqrt{2} \tag{1.4}$$

$$\sum_{k=0}^{\infty}(-1)^k\binom{2k}{k}\frac{1}{2^{2k}}=\frac{\sqrt{2}}{2} \tag{1.5}$$

$$-\sum_{k=0}^{\infty}\binom{2k}{k}\frac{x^k}{2^{2k}(2k-1)}=\sqrt{1-x}, \qquad |x|<1 \tag{1.6}$$

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{2^{2k}(2k-1)}=0 \tag{1.7}$$

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{(-1)^k}{2^{2k}(2k-1)}=-\sqrt{2} \tag{1.8}$$

## 1.2 Companion Binomial Theorem

**Remark 1.2** *In Section 1.2, we assume $p$ is a nonnegative integer. We also assume $a$ and $b$ are arbitrary real or complex numbers.*

*Companion Binomial Theorem*

$$\boldsymbol{\sum_{n=0}^{\infty}\binom{n+p}{n}x^n=\sum_{n=p}^{\infty}\binom{n}{p}x^{n-p}=\frac{1}{(1-x)^{p+1}}, \qquad |x|<1} \tag{1.9}$$

### 1.2.1 Applications of Companion Binomial Theorem

$$\frac{1}{(a+b)^{p+1}}=\frac{1}{a^{p+1}}\sum_{k=0}^{\infty}(-1)^k\binom{k+p}{k}\frac{b^k}{a^k}, \qquad |\frac{b}{a}|<1 \tag{1.10}$$

$$\frac{1}{(a-b)^{p+1}}=\frac{1}{a^{p+1}}\sum_{k=0}^{\infty}\binom{k+p}{k}\frac{b^k}{a^k}, \qquad |\frac{b}{a}|<1 \tag{1.11}$$

$$\frac{1}{(1+x^m)^{p+1}}=\sum_{k=0}^{\infty}(-1)^k\binom{k+p}{k}x^{mk}, \qquad |x|<1,\ m\in\Re \tag{1.12}$$

$$\frac{1}{(1-x^m)^{p+1}} = \sum_{k=0}^{\infty} \binom{k+p}{k} x^{mk}, \qquad |x| < 1,\ m \in Re \tag{1.13}$$

$$\frac{1}{(x^m+1)^{p+1}} = \sum_{k=0}^{\infty} (-1)^k \binom{k+p}{k} x^{-m(k+p+1)}, \qquad |x| > 1,\ m \in \Re \tag{1.14}$$

$$\frac{1}{(x^m-1)^{p+1}} = \sum_{k=0}^{\infty} \binom{k+p}{k} x^{-m(k+p+1)}, \qquad |x| > 1,\ m \in \Re \tag{1.15}$$

### 1.2.2 Finite Version of Companion Binomial Theorem

$$\sum_{k=0}^{n} \binom{n+k}{k} \frac{1}{2^k} = 2^n \tag{1.16}$$

*Variation of Finite Companion Binomial Theorem*

$$\sum_{k=0}^{n} \binom{2n-k}{n} 2^k = 2^{2n} \tag{1.17}$$

*Application of Finite Companion Binomial Theorem*

$$\sum_{k=1}^{\infty} \binom{2n+k}{n} \frac{1}{2^k} = 2^{2n} \tag{1.18}$$

## 1.3 Binomial Theorem with Complex Exponents

**Remark 1.3** *The material in Section 1.3 is found in T. J. I'a. Bromwich's Introduction to the Theory of Infinite Series, Second Edition, 1949, Chapter 9, Article 96.*

Let $\alpha, \beta \in \Re$. Let $i^2 \equiv -1$. Then,

$$\boldsymbol{(1+x)^{\alpha+\beta i} = \sum_{k=0}^{\infty} \binom{\alpha+\beta i}{k} x^k,} \tag{1.19}$$

*where*

a. *The series is absolutely convergent for* $|x| < 1$.

b. *If* $\alpha > 0$*, the series converges absolutely on the circle* $|x| = 1$*. Hence, the series is uniformly convergent within and on the circle* $|x| = 1$.

c. *If* $-1 < \alpha \leq 0$*, the series converges on the circle* $|x| = 1$ *except at* $x = -1$.

d. *If* $\alpha \leq -1$*, the series diverges everywhere on the circle* $|x| = 1$.

## 1.4 Applications of the Binomial Theorem

### 1.4.1 Derivatives of the Binomial Series

$$\boldsymbol{\sum_{k=0}^{n} \binom{n}{k} \binom{k}{r} x^k = x^r (1+x)^{n-r} \binom{n}{r}} \tag{1.20}$$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{k}{r} = 2^{n-r} \binom{n}{r} \tag{1.21}$$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{k}{r} 2^k = 2^r 3^{n-r} \binom{n}{r} \tag{1.22}$$

$$\boldsymbol{\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k}{r} x^k = (-1)^r x^r (1-x)^{n-r} \binom{n}{r}} \tag{1.23}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k 2^k = 2n(-1)^n \tag{1.24}$$

### 1.4.2 Expansions of $(1-x)^{-\frac{1}{2}}$

$$\sum_{k=0}^{\infty}(-1)^k\binom{\frac{-1}{2}}{k}x^k = \frac{1}{\sqrt{1-x}}, \qquad |x|<1 \tag{1.25}$$

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{x^k}{(1+x)^{2k+1}} = \frac{1}{1-x}, \qquad |x|<1 \text{ or } |\frac{x}{1+x}|<1 \tag{1.26}$$

$$\sum_{k=0}^{\infty}\binom{2k}{k}\binom{k}{r}\frac{x^k}{2^{2k}} = (1-x)^{-\frac{2r+1}{2}}\left(\frac{x}{4}\right)^r\binom{2r}{r}, \qquad |x|<1 \tag{1.27}$$

$$\sum_{k=0}^{\infty}\binom{2k}{k}\binom{k}{r}\frac{1}{2^{3k}} = \frac{\sqrt{2}}{2^{2r}}\binom{2r}{r} \tag{1.28}$$

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{k}{2^{3k}} = \frac{\sqrt{2}}{2} \tag{1.29}$$

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{k}{2^{2k}}x^k = \frac{x}{2(1-x)^{\frac{3}{2}}}, \qquad |x|<1 \tag{1.30}$$

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{2k+1}{2^{2k}}x^k = \frac{1}{(1-x)^{\frac{3}{2}}}, \qquad |x|<1 \tag{1.31}$$

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{2k+1}{2^{3k}} = \sqrt{8} \tag{1.32}$$

*Bruckman's Formula Version 1*

$$\sum_{k=0}^{n}\binom{\frac{-1}{2}}{k}\binom{\frac{-1}{2}}{n-k}\frac{1}{(2k+1)(2n-2k+1)} = \frac{-1}{2(n+1)^2\binom{\frac{-1}{2}}{n+1}} \tag{1.33}$$

*Bruckman's Formula Version 2*

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{(2k+1)(2n-2k+1)} = \frac{2^{4n+1}}{(n+1)^2\binom{2n+2}{n+1}} = \frac{2^{4n}}{(n+1)(2n+1)\binom{2n}{n}} \tag{1.34}$$

### 1.4.3 Expansions of $(1-x)^{\frac{1}{2}}$

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{x^k}{4^k(2k-1)} = -(1-x)^{\frac{1}{2}}, \qquad |x|<1 \tag{1.35}$$

$$\sum_{k=0}^{\infty}\binom{2k+1}{k}\frac{z^{k+1}}{2k+1} = \frac{1-(1-4z)^{\frac{1}{2}}}{2}, \qquad |z|<\frac{1}{4} \tag{1.36}$$

### 1.4.4 Evaluation of $\sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{k+1}$

$$\boldsymbol{\sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{k+1} = \frac{(x+1)^{n+1}-1}{(n+1)x}, \qquad x\neq 0} \tag{1.37}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1} = \frac{2^{n+1}-1}{n+1} \tag{1.38}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{k+1} = \frac{1}{n+1} \tag{1.39}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\frac{2^k}{k+1} = \frac{1}{2n+1} \tag{1.40}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n+1}{k}\frac{2^k}{k+1} = \frac{2^{2n}}{n+1} \tag{1.41}$$

$$\sum_{k=1}^{2n}(-1)^k\binom{2n}{k-1}\frac{2^k}{k+1} = \frac{2^{2n}}{n+1}-\frac{1}{2n+1}, \qquad n\geq 1 \tag{1.42}$$

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}\frac{x^{2k}}{2k+1} = \frac{(x+1)^{n+1}-(1-x)^{n+1}}{2(n+1)x}, \qquad x\neq 0 \tag{1.43}$$

$$\sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} \frac{1}{2k+1} = \frac{2^n}{n+1} \tag{1.44}$$

$$\sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} \frac{x^{2k}}{k+1} = \frac{(x+1)^{n+1} + (1-x)^{n+1} - 2}{(n+1)x^2}, \qquad x \neq 0 \tag{1.45}$$

$$\sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} \frac{1}{k+1} = \frac{2^{n+1} - 2}{n+1} \tag{1.46}$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(k+r)!} x^k = \frac{n!\left((x+1)^{n+r} - \sum_{k=0}^{r-1} \binom{n+r}{k} x^k\right)}{(n+r)!x^r}, \qquad x \neq 0,\ r \geq 1 \tag{1.47}$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{x^{k+r}}{\binom{k+r}{k}} = \frac{(x+1)^{n+r} - \sum_{i=0}^{r-1} \binom{n+r}{i} x^i}{\binom{n+r}{n}}, \qquad r \geq 1 \tag{1.48}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k!}{(k+r)!} = (-1)^{r+1} n! \sum_{k=0}^{r-1} (-1)^k \frac{1}{(n+r-k)!k!}, \qquad r \geq 1 \tag{1.49}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{\binom{k+r}{k}} \left( (1-x)^{k+r} - \sum_{\alpha=0}^{r-1} (-1)^\alpha \binom{k+r}{\alpha} x^\alpha \right) = (-1)^r \frac{x^{n+r}}{\binom{n+r}{n}},\ r \geq 1 \tag{1.50}$$

### 1.4.5 Expansions of $(t-a)^{n-1}(t-(a+nx))$

**Remark 1.4** *In the identities related to the expansion of* $(t-a)^{n-1}(t-(a+nx))$ *, we assume, unless otherwise specified, that* $x$*,* $t$*, and* $a$ *are nonzero real or complex numbers.*

$$\boldsymbol{\sum_{k=0}^{n} (-1)^k \binom{n}{k} t^{n-k} a^{k-1} (a+kx) = (t-a)^{n-1}(t-(a+nx)), \qquad n \geq 1} \tag{1.51}$$

$$\lim_{a \to 0} \sum_{k=0}^{n} (-1)^k \binom{n}{k} t^{n-k} a^{k-1} (a+kx) = t^n - nxt^{n-1}, \qquad n \geq 1 \tag{1.52}$$

$$a^{n-1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(a+kx)=\begin{cases}0, & n\neq 0,1\\ -x, & n=1\\ 1, & n=0\end{cases} \tag{1.53}$$

$$\boldsymbol{\sum_{k=0}^{n}(-1)\binom{n}{k}(f(x))^k(a+kx)= (1-f(x))^{n-1}(a-(a+nx)f(x)), \qquad n\geq 1} \tag{1.54}$$

$$\boldsymbol{\sum_{k=0}^{n}(-1)\binom{n}{k}(f(x))^k(a-kx)= (1-f(x))^{n-1}(a-(a-nx)f(x)), \; n\geq 1} \tag{1.55}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a+kx}{n^k}=\left(1-\frac{1}{n}\right)^{n-1}\left(a-\left(\frac{a}{n}+x\right)\right), \qquad n\geq 1 \tag{1.56}$$

$$\lim_{n\to\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a+kx}{n^k}=\frac{a-x}{e} \tag{1.57}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{a+kx}{n^k}=\left(1+\frac{1}{n}\right)^{n-1}\left(a+\left(\frac{a}{n}+x\right)\right), \qquad n\geq 1 \tag{1.58}$$

$$\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{a+kx}{n^k}=(a+x)e \tag{1.59}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a+kx}{2^k}=\frac{a-nx}{2^n} \tag{1.60}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a+kx}{2^{nk}}=\left(1-\frac{1}{2^n}\right)^{n-1}\left(a-\frac{a+nx}{2^n}\right), \qquad n\geq 1 \tag{1.61}$$

$$\lim_{n\to\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a+kx}{2^{nk}}=a \tag{1.62}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a+kx}{(a+nx)^k}=\left(1-\frac{1}{a+nx}\right)^{n-1}(a-1),\qquad n\geq 1 \tag{1.63}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a+kx}{(a-nx)^k}=\left(1-\frac{1}{a-nx}\right)^{n-1}\left(a-\frac{a+nx}{a-nx}\right),\qquad n\geq 1 \tag{1.64}$$

$$\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}k\left(f(x)\right)^k=nf(x)\left(1-f(x)\right)^{n-1} \tag{1.65}$$

$$\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}k^2\left(f(x)\right)^{k-1}=n\left(1-f(x)\right)^{n-1}-n(n-1)f(x)\left(1-f(x)\right)^{n-2} \tag{1.66}$$

$$\sum_{k=0}^{n}\binom{n}{k}k^2=2^n\left(\frac{n^2+n}{4}\right) \tag{1.67}$$

$$\sum_{k=0}^{n}\binom{n}{k}k^2 2^{k-1}=3^{n-2}n(2n+1) \tag{1.68}$$

$$\sum_{k=0}^{n}\binom{n}{k}a^k k=na(1+a)^{n-1} \tag{1.69}$$

$$\sum_{k=0}^{n}\binom{n}{k}k=n2^{n-1} \tag{1.70}$$

$$\sum_{k=0}^{n}\binom{n}{k}k(k-1)x^{k-2}=n(n-1)(1+x)^{n-2} \tag{1.71}$$

$$\sum_{k=0}^{n}\binom{n}{k}k(k-1)=n(n-1)2^{n-2} \tag{1.72}$$

### 1.4.6 Number Theoretic Result Due to Euler

$$\text{Let } f(x) = \sum_{i=0}^{n} a_i x^i. \qquad \text{Then, } f(x)|f(x+f(x)). \tag{1.73}$$

## 1.5 Four Versions of the Multinomial Theorem

**Remark 1.5** *In Section 1.5, we will assume $\alpha$ is a nonnegative integer. We also assume that $j_i$ is a nonnegative integer.*

$$\left(\sum_{i=0}^{n} \boldsymbol{a_i}\right)^{\boldsymbol{\alpha}} = \sum_{\substack{\forall \boldsymbol{j} \text{such that} \\ \sum_{i=0}^{n} \boldsymbol{j_i}=\boldsymbol{\alpha}}} \frac{\boldsymbol{\alpha}!}{\boldsymbol{j_0! j_1! j_2! \ldots j_n!}} \boldsymbol{a_0^{j_0} a_1^{j_1} \ldots a_n^{j_n}} \tag{1.74}$$

$$\left(\sum_{i=1}^{n} a_i\right)^{\alpha} = \sum_{\substack{\forall j \text{such that} \\ \sum_{i=1}^{n} j_i=\alpha}} \frac{\alpha!}{j_1! j_2! \ldots j_n!} a_1^{j_1} a_2^{j_2} \ldots a_n^{j_n} \tag{1.75}$$

$$\left(\sum_{i=0}^{n} \boldsymbol{a_i x^i}\right)^{\boldsymbol{\alpha}} = \sum_{k=0}^{N<\infty} \boldsymbol{x^k} \sum_{\substack{\forall \boldsymbol{j} \text{such that} \\ \sum_{i=0}^{n} \boldsymbol{j_i}=\boldsymbol{\alpha},\ \sum_{i=1}^{n} \boldsymbol{i j_i}=\boldsymbol{k}}} \frac{\boldsymbol{\alpha}!}{\boldsymbol{j_0! j_1! j_2! \ldots j_n!}} \boldsymbol{a_0^{j_0} a_1^{j_1} \ldots a_n^{j_n}} \tag{1.76}$$

$$\left(\sum_{i=0}^{n} \boldsymbol{a_i x^i}\right)^{\boldsymbol{\alpha}} = \sum_{k=0}^{N<\infty} \boldsymbol{x^k} *$$

$$\sum_{\substack{\forall \boldsymbol{j} \text{such that} \\ \boldsymbol{j_0}+\gamma=\boldsymbol{\alpha},\ \sum_{i=1}^{n} \boldsymbol{j_i}=\gamma,\ \sum_{i=1}^{n} \boldsymbol{i j_i}=\boldsymbol{k}}} \binom{\boldsymbol{\alpha}}{\gamma} \boldsymbol{a_0^{\alpha-\gamma}} \frac{\gamma!}{\boldsymbol{j_1! j_2! \ldots j_n!}} \boldsymbol{a_1^{j_1} a_2^{j_2} \ldots a_n^{j_n}} \tag{1.77}$$

# 2 The Geometric Series

**Remark 2.1** *In this chapter, we will assume, unless otherwise specified, that $a$ is a nonnegative integer and $x$ is an arbitrary nonzero real or complex number.*

## 2.1 The Basic Geometric Series

### 2.1.1 Finite Geometric Series

$$\boldsymbol{\sum_{k=a}^{n} x^k = x^a \frac{x^{n-a+1}-1}{x-1}, \qquad x \neq 1} \tag{2.1}$$

$$\sum_{k=a}^{n} \frac{1}{x^k} = \frac{1}{x^n}\frac{x^{n-a+1}-1}{x-1}, \qquad x \neq 1 \tag{2.2}$$

### 2.1.2 Infinite Geometric Series

$$\boldsymbol{\sum_{k=a}^{\infty} x^k = \frac{x^a}{1-x}, \qquad |x| < 1} \tag{2.3}$$

$$\sum_{k=a}^{\infty} x^{-k} = \frac{x^{1-a}}{x-1}, \qquad |x| > 1 \tag{2.4}$$

## 2.2 Derivatives of Geometric Series

$$\sum_{k=0}^{n} kx^{k-1} = \frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}, \qquad x \neq 1 \tag{2.5}$$

$$\lim_{x\to 1}\sum_{k=0}^{n} kx^k = \frac{n^2+n}{2} = \sum_{k=0}^{n} k \tag{2.6}$$

$$\sum_{k=0}^{n} k^2x^k = \frac{n^2x^{n+3}-(2n^2+2n-1)x^{n+2}+(n+1)^2x^{n+1}-x^2-x}{(x-1)^3}, \qquad x \neq 1 \tag{2.7}$$

$$\sum_{k=0}^{n} k(k-1)2^k = (n^2-3n+4)2^{n+1}-2^3 \tag{2.8}$$

$$\sum_{k=0}^{n} k^2 2^k = (n^2 - 2n + 3)2^{n+1} - 6 \tag{2.9}$$

$$\sum_{k=0}^{n} k^2 3^k = \frac{(n^2 - n + 1)3^{n+1} - 3}{2} \tag{2.10}$$

## 2.3 Integrals of Geometric Series

$$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}, \qquad |x| < 1 \tag{2.11}$$

$$\ln 2 = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} \tag{2.12}$$

## 2.4 Applications of Geometric Series

**Remark 2.2** *In the following two identities, let $u$ and $v$ be arbitrary nonzero real or complex numbers such that $uv \neq 1$.*

$$\sum_{k=0}^{[\frac{n-1}{2}]} (uv)^k + u \sum_{k=0}^{[\frac{n-2}{2}]} (uv)^k = \frac{(uv)^{[\frac{n+1}{2}]} - 1}{uv - 1} + u \frac{(uv)^{[\frac{n}{2}]} - 1}{uv - 1}, \qquad n \geq 1 \tag{2.13}$$

**Remark 2.3** *The following identity can be done as a formal calculation over the ring of power series. Otherwise, the reader may assume that appropriate condition hold so that the left sum is absolutely convergent.*

$$\boldsymbol{\sum_{k=0}^{\infty} f(k) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \sum_{i=0}^{k} 2^i f(i)} \tag{2.14}$$

$$\sum_{k=0}^{\infty} \frac{x^{2^k}}{1 - x^{2^{k+1}}} = \frac{1}{1-x}, \qquad |x| > 1 \tag{2.15}$$

$$\sum_{k=0}^{\infty} \frac{x^{2^k}}{1 - x^{2^{k+1}}} = \frac{x}{1-x}, \qquad |x| < 1 \tag{2.16}$$

# 3 Bernoulli-Type Series and the Riemann Zeta Function

**Remark 3.1** *In this chapter, we will assume $p$ and $a$ are, unless otherwise specified, nonnegative integers.*

## 3.1 Evaluation of $\sum_{k=1}^{n} k^p$

### 3.1.1 Reduction Formula

$$\sum_{k=a}^{n} k^p = n\sum_{k=a}^{n} k^{p-1} - \sum_{r=a}^{n-1}\sum_{k=a}^{r} k^{p-1}, \qquad p \geq 1,\ \ n \geq 1 \tag{3.1}$$

### 3.1.2 Iteration Formulas

**Remark 3.2** *In this subsection, we let $r$ be a positive integer, and define $\sum_{(r)}^{n} f(k)$ to be the following $r-$fold sum:*

$$\sum_{(r)}^{n} f(k) = \sum_{k_r=1}^{n}\sum_{k_{r-1}=1}^{k_r} \cdots \sum_{k_2=1}^{k_3}\sum_{k=1}^{k_2} f(k) >$$

$$\sum_{(r)}^{n} 1 = \binom{n+r-1}{r} \tag{3.2}$$

$$\sum_{(r)}^{n} k = \binom{n+r}{r+1} \tag{3.3}$$

$$\sum_{(1)} k = \frac{n(n+1)}{2} \tag{3.4}$$

$$\sum_{(r)}^{n} k^2 = \frac{(2n+r)}{(r+2)!}\frac{(n+r)!}{(n-1)!} \tag{3.5}$$

$$\sum_{(1)}^{n} k^2 = \frac{n(n+1)(2n+1)}{3!} \tag{3.6}$$

$$\sum_{(r)}^{n} k^3 = \frac{6n^2 + r(6n + r - 1)}{(r+3)!} \frac{(n+r)!}{(n-1)!} \tag{3.7}$$

$$\sum_{(1)}^{n} k^3 = \frac{n^2(n+1)^2}{4} \tag{3.8}$$

$$\sum_{(r)}^{n} k^4 = \frac{(12n^2 + 12rn - r(5-r))(2n+r)}{(r+4)!} \frac{(n+r)!}{(n-1)!} \tag{3.9}$$

$$\sum_{(1)}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \tag{3.10}$$

$$\sum_{(1)}^{n} k^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12} \tag{3.11}$$

$$\sum_{(1)}^{n} k^6 = \frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42} \tag{3.12}$$

### 3.1.3 Euler's Expansion with Bernoulli Numbers

**Remark 3.3** *In this subsection, we let $\mathcal{B}_k$ denote the $k^{th}$ Bernoulli number. The exponential generating function of $(\mathcal{B}_k)_{n=0}^{\infty}$ is $\frac{x}{e^x-1}$. For values of the Bernoulli number sequence, the reader is referred to the Online Encyclopedia of Integer Sequences (OEIS).*

$$\boldsymbol{\sum_{k=0}^{n-1} k^p = \sum_{k=0}^{p} \frac{p!n^{p-k+1}}{(p-k+1)!k!}\mathcal{B}_k, \qquad n \geq 1} \tag{3.13}$$

### 3.1.4 G. P. Miller's Determinant Expansion

$$\sum_{k=0}^{n} k^p = \frac{\det M}{(p+1)!}, \tag{3.14}$$

where $M$ is the $(p+1) \times (p+1)$ matrix whose entry $a_{i,j}$ is determined by

$$a_{i,j} = \begin{cases} (n+1)^{p+2-i} - (n+1), & j = 1 \\ \binom{p+2-i}{j+1-i}, & j \geq i \text{ and } j \neq 1 \\ 0, & j < i \text{ and } j \neq 1 \end{cases} \tag{3.15}$$

## 3.2 Evaluation of $\sum_{k=0}^{n} k^p x^k$

**Remark 3.4** *The reader should compare the formulas in this subsection with those in Section 2.2.*

### 3.2.1 Differential Reduction Formula

$$\sum_{k=0}^{n} k^{p+1} x^k = x \frac{d}{dx} \sum_{k=0}^{n} k^p x^k \tag{3.16}$$

### 3.2.2 Applications of Differential Reduction Formula

$$\sum_{k=0}^{n} k^3 x^k = \frac{N}{(x-1)^4}, \qquad \text{where for } x \neq 1, \tag{3.17}$$

$N = n^3 x^{n+4} - (3n^3 + 3n^2 - 3n + 1)x^{n+3} + (3n^3 + 6n^2 - 4)x^{n+2} - (n+1)^3 x^{n+1} + x^3 + 4x^2 + x$

$$\sum_{k=1}^{n-1} k \left(\frac{n}{n-1}\right)^{k-1} = (n-1)^2, \qquad n \geq 1 \tag{3.18}$$

$$\prod_{k=2}^{n} \left(1 + \frac{\left(\frac{n+1}{n}\right)^{k-1}}{k - \left(\frac{k+1}{k}\right)^{k-2}}\right) = n^2, \qquad n \geq 2 \tag{3.19}$$

$$\sum_{k=1}^{n} \frac{k}{n^{k-1}} (n+1)^{k-1} = \sum_{k=1}^{n} \frac{k}{n^{k-1}} \binom{n+1}{k+1} = n^2 \qquad n \geq 1 \tag{3.20}$$

## 3.3 Evaluation of $\sum_{k=0}^{n} \binom{n}{k} k^p$

$$\boldsymbol{\sum_{r=0}^{n} \binom{n}{r} = 2^n} \tag{3.21}$$

$$\boldsymbol{\sum_{r=0}^{n} \binom{n}{r} r = n 2^{n-1}} \tag{3.22}$$

$$\sum_{r=0}^{n} \binom{n}{r} r^2 = 2^{n-2} n(n+1) \tag{3.23}$$

$$\sum_{r=0}^{n}\binom{n}{r}r^3 = 2^{n-3}n^2(n+3) \tag{3.24}$$

$$\sum_{r=0}^{n}\binom{n}{r}r^4 = 2^{n-4}n(n+1)(n^2+5n-2) \tag{3.25}$$

### 3.3.1 Reduction Formula

$$n\sum_{r=0}^{n-1}\binom{n-1}{r}r^p = \sum_{k=0}^{p}\sum_{r=0}^{n}(-1)^k\binom{p}{k}\binom{n}{r}r^{p-k+1}, \qquad n \geq 1 \tag{3.26}$$

## 3.4 Riemann Zeta Function: $\zeta(p) = \sum_{k=1}^{\infty}\frac{1}{k^p}$

### 3.4.1 Convolution Identity

**Remark 3.5** *The following identity is found in "A New Method of Evaluating $\zeta(2n)$", by G.T. Williams, Amer. Math. Monthly, January 1953, Vol. 60, No. 1, pp. 19-25.*

$$\sum_{k=2}^{n-1}\zeta(k)\zeta(n-k+1) = (n+2)\zeta(n+1) - 2\sum_{k=1}^{\infty}\frac{1}{k^n}\sum_{k=1}^{k}\frac{1}{j}, \qquad n \geq 3 \tag{3.27}$$

*Extension of Convolution Identity*

$$4\zeta(3) - 2\sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{j=1}^{k}\frac{1}{j} = 0 \tag{3.28}$$

### 3.4.2 Connections with Bernoulli Numbers

**Remark 3.6** *Recall that $\mathcal{B}_n$ is the $n^{th}$ Bernoulli number. See Remark 3.3.*

$$\mathcal{B}_{2n} = (-1)^{n-1}\frac{(2n)!}{2^{2n-1}\pi^{2n}}\zeta(2n) \tag{3.29}$$

$$\sum_{k=1}^{n-1}\zeta(2k)\zeta(2n-2k) = \left(n+\frac{1}{2}\right)\zeta(2n), \qquad n \geq 2 \tag{3.30}$$

# 4 Finite Harmonic Series

## 4.1 Special Case of $n^{th}$ Difference Inversion Formula

$$\boldsymbol{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{k+1}=\frac{1}{n+1}} \tag{4.1}$$

$$\sum_{j=1}^{n}\frac{1}{j}=\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\frac{1}{k}, \qquad n\geq 1 \tag{4.2}$$

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\sum_{j=1}^{k}\frac{1}{j}=\frac{1}{n}, \qquad n\geq 1 \tag{4.3}$$

## 4.2 Even and Odd Finite Harmonic Series

$$\sum_{k=1}^{2n}\frac{1}{k}=\frac{2n+1}{2}\sum_{k=1}^{2n}\frac{1}{k(2n-k+1)}=(2n+1)\sum_{k=1}^{n}\frac{1}{k(2n-k+1)}, \qquad n\geq 1 \tag{4.4}$$

$$\sum_{k=1}^{2n+1}\frac{1}{k}=(2n+2)\sum_{k=1}^{n}\frac{1}{k(2n-k+2)}+\frac{1}{n+1} \tag{4.5}$$

## 4.3 Harmonic Series as Limit of Binomial Coefficient

**Remark 4.1** *In the following indentity, we let $r$ be a positive integer.*

$$\sum_{k=1}^{n}\frac{1}{k}=\lim_{r\to 0}\frac{\binom{n+r}{r}-1}{r\binom{n+r}{r}} \tag{4.6}$$