Some Symmetry and Unimodality Properties of the q, x, y-Hit Numbers

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Abstract

We prove symmetry and in some cases symmetry and unimodality of polynomials related to the q, x, y-hit numbers introduced by Haglund. These results generalize theorems proven by Haglund for the q-hit numbers. We also apply one of these results to obtain a corollary concerning a generalization of the Eulerian numbers.

Résumé

Nous prouvons la symétrie et dans certains cas la symétrie et l'unimodalité de polynômes relatifs aux q, x, y nombres de contacts introduits par Haglund, généralisant ainsi certains théorèmes. Un de ces résultats nous permet d'obtenir un corollaire à propos d'une généralisation des nombres Eulériens.

1 Notation

Definition 1.1. *A* ***regular Ferrers board*** *is a Ferrers board (a board with contiguous columns of non-decreasing heights from left to right) with the additional property that the ith column has height at least i.*

SQ_n denotes the $n \times n$ square chess board. We use $B(b_1, \ldots, b_n)$ to denote the Ferrers board with column heights $b_1 \leq b_2 \leq \cdots \leq b_n$, and $B(h_1, d_1; \ldots; h_t, d_t)$ to denote the Ferrers board with step heights $h_1, \ldots h_t$ and step depths $d_1, \ldots d_t$.

If $B = B(h_1, d_1; \ldots; h_t, d_t) \subseteq SQ_n$ is a Ferrers board, let us denote by $B - h_p - d_p$ the Ferrers board $B(h_1, d_1; \ldots; h_p - 1, d_p - 1; \ldots h_t, d_t) \subseteq SQ_{n-1}$, obtained from B by decreasing the pth step by 1.

We use the usual notation for common *q-analogs*:

$$[z] = \frac{1 - q^z}{1 - q} \text{ for } z \in \mathbb{R}$$

$$[n] = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1} \text{ for } n \in \mathbb{N}$$

$$[n]! = [n][n-1] \cdots [2][1]$$

$$\begin{bmatrix} z \\ k \end{bmatrix} = \frac{[z][z-1] \cdots [z-k+1]}{[k]!} \text{ for } z \in \mathbb{R}, k \in \mathbb{N}.$$

2 Definition of $A_{n,k}(x,y,q,B)$

We prove symmetry and in some cases unimodality of polynomials related to the *q,x,y-hit numbers* $A_{n,k}(x,y,q,B)$ of Haglund [8], for B a regular Ferrers board. These polynomials, which generalize many previously studied hit numbers, were defined algebraically via the equation

$$\sum_{k=0}^{n} A_{n,k}(x,y,q,B) z^k =$$
$$\sum_{k=0}^{n} R_{n-k}(y,q,B)[x][x+1]\cdots[x+k-1]z^k \prod_{i=k+1}^{n}(1-zq^{x+i-1}).$$

3 Definition of $R_k(y,q,B)$

Here $R_k(y,q,B)$ is the *kth cycle-counting q-rook number* of Ehrenborg, Haglund, and Readdy [4], which generalize many previously studied rook numbers. It is defined by the equation

$$R_k(y,q,B) = \sum_{P\ k \text{ rooks on } B} [y]^{cyc(P)} q^{inv_B(P)+(y-1)E(P)}.$$

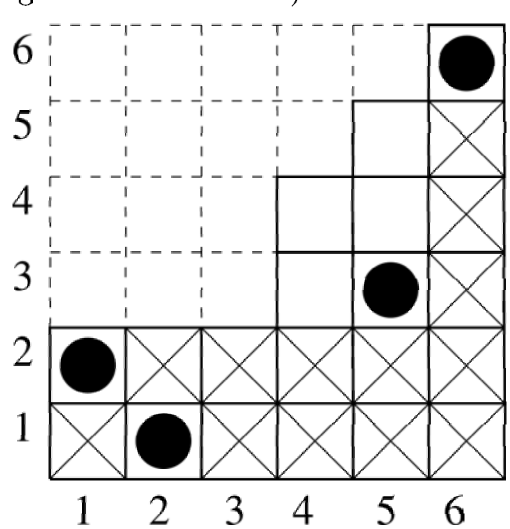
For P a placement of rooks on a Ferrers board $B = B(b_1,\ldots,b_n) \subseteq SQ_n$, we have the following.

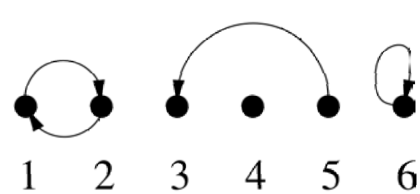
1. $inv_B(P)$: Let each rook cancel all squares to the right in its row and below in its column; then define $inv_B(P)$ to be the number of squares of B which neither contain a rook from P nor are cancelled (see Garsia and Remmel [5]).

2. $cyc(P)$: We can associate to P a simple directed graph G_P on n vertices, where there is an edge from i to j in G_P if and only if there is a rook from P on the square (i,j). We can then define $cyc(P)$ to be the number of cycles in G_P (see Gessel [6], Chung and Graham [2] or Dworkin [3]).

3. $E(P)$: This statistic is the number of i such that there is no rook from P in column i of B on or above square $s_i(P)$. Here $s_i(P)$ is the unique square on which, considering only rooks from P in columns 1 through $i-1$ of P, placing a rook would complete a new cycle in the associated digraph (see Haglund [8]).

For the rook placement P pictured below, we see that $inv_B(P) = 4$, $cyc(P) = 2$, and $E(P) = 2$ (corresponding to $i = 4$ and $i = 5$).





4 Symmetry and unimodality of $A_{n,k}(a,b,q,B)$

Definition 4.1. *Suppose*

$$f(q) = \sum_{i=M}^{N} a_i q^i$$

*is a polynomial in q with $a_M, a_N \neq 0$. We call $M+N$ the **virtual degree** of f. We will say the polynomial $f(q)$ is **zsu(d)** if either $f(q)$ is identically zero, or $f(q)$ is in $\mathbb{N}[q]$, symmetric, and unimodal with virtual degree d.*

We use the following lemmas to prove Corollary 4.5 below. The proof of Lemma 4.2 is trivial, and a proof of Lemma 4.3 can be found in [9].

Lemma 4.2. *If f and g are polynomials which are both $zsu(d)$, then $f+g$ is $zsu(d)$.*

Lemma 4.3. *If f is $zsu(d)$ and g is $zsu(e)$, then fg is $zsu(d+e)$.*

Lemma 4.4. *Let $B = B(h_1, d_1; \ldots; h_t, d_t) \subseteq SQ_n$ be a regular Ferrers board, $B - h_t - d_t \subseteq SQ_{n-1}$ as described earlier. Then*

$$A_{n,k}(x,y,q,B) = [k+y+d_t-1]A_{n-1,k}(x,y,q,B-h_t-d_t)+$$

$$q^{k+y+d_t-2}[n+x-y-d_t-k+1]A_{n-1,k-1}(x,y,q,B-h_t-d_t)$$

for any $1 \leq k \leq n$.

Proof. Let $p = t$ in Lemma 5.7 of [8]. □

The following is now a simple corollary of the above lemmas.

Corollary 4.5. *Let $B = B(h_1, d_1; \ldots; h_t, d_t) \subseteq SQ_n$ be a regular Ferrers board, $a, b \in \mathbb{N}$. If $n+a+1 \geq b+d_t+k$, then $A_{n,k}(a,b,q,B)$ is $zsu(Area(B) + n(b+k-1) + k(a-1) - \binom{n+1}{2})$ for $0 \leq k \leq n$.*

Proof. The proof is by induction on $Area(B)$, using the above lemmas. □

5 Symmetry and unimodality of the cycle-counting q-Eulerian numbers

Definition 5.1. *The number of **left-to-right minima** of a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$, denoted $\ell rmin(\sigma)$, is computed by the following algorithm.*

1. *Find j_1 such that $\sigma_{j_1} = 1$.*
2. *Let y_1 be the cycle $(\sigma_1 \cdots \sigma_{j_1})$.*
3. *Find α which is the smallest integer not contained in y_1.*
4. *Find j_2 such that $\sigma_{j_2} = \alpha$.*
5. *Let y_2 be the cycle $(\sigma_{j_1+1} \cdots \sigma_{j_2})$.*
6. *Continue in this manner until σ is partitioned into cycles $y_1y_2\cdots y_p$; then $p = \ell rmin(\sigma)$.*

Definition 5.2. *The* k***th cycle-counting*** q***-Eulerian number*** *is defined by the equation*

$$\tilde{E}_{n,k}(y,q) = \sum_{\sigma \in S_n,\ des(\sigma)=k-1} [y]^{\ell rmin(\sigma)} q^{(n-\ell rmin(\sigma))(y-1)+maj(\sigma)},$$

where $maj(\sigma)$ *denotes the* ***major index*** *of* σ.

It was proven in [1] that

$$\tilde{E}_{n,k}(y,q) = A_{n,k-1}(y,y,q,\mathbb{T}_n), \tag{1}$$

where $\mathbb{T}_n = B(1,2,\ldots,n)$ denotes the triangular Ferrers board. In light of (1) and Corollary 4.5, the following can be easily proven.

Corollary 5.3. *For* $m \in \mathbb{N}$, *the polynomial*

$$\sum_{\sigma \in S_n,\ des(\sigma)=k-1} [m]^{\ell rmin(\sigma)} q^{(n-\ell rmin(\sigma))(m-1)+maj(\sigma)}$$

is symmetric and unimodal with virtual degree $n(m+k-2)+(k-1)(m-1)$.

6 Symmetry of $A_{n,k}(a,b,q,B)/\prod_{i=1}^{t}[d_i]!$

Definition 6.1. *For* $B = B(h_1, d_1; \ldots; h_t, d_t)$, *we will use the notation* H_i *for the partial sum* $h_1 + \cdots + h_i$, *and* D_i *for* $d_1 + \cdots + d_i$.

Lemma 6.2. *Let* $B = B(h_1, d_1; \ldots; h_t, d_t) \subseteq SQ_n$ *be a regular Ferrers board. Then*

$$\frac{A_{n,k}(x,y,q,B)}{\prod_{i=1}^{t}[d_i]!} =$$

$$\sum_{j=0}^{k} \begin{bmatrix} n+x \\ k-j \end{bmatrix} \begin{bmatrix} x+j-1 \\ j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} \prod_{i=1}^{t} \begin{bmatrix} j+H_i-D_{i-1}+y-1 \\ d_i \end{bmatrix}.$$

Proof. By Lemma 5.1 of [8]. □

We can now prove the following.

Theorem 6.3. *Let* $B = B(h_1, d_1; \ldots; h_t, d_t) \subseteq SQ_n$ *be a regular Ferrers board,* $a, b \in \mathbb{N}$ *with* $a \geq b \geq 1$, *and set*

$$L_k^{a,b}(B) = Area(B) + n(b-1) + k(n+a-1) - \sum_{i=1}^{t} d_i D_i.$$

Then $A_{n,k}(a,b,q,B)/\prod_{i=1}^{t}[d_i]!$ *is either zero or symmetric with virtual degree* $L_k^{a,b}(B)$.

Proof. By Lemma 6.2, we get an explicit formula for $A_{n,k}(a,b,q,B)/\prod_{i=1}^{t}[d_i]!$ as a sum of products of q-binomial coefficients and powers of q. Using the fact that $\begin{bmatrix} r \\ s \end{bmatrix}$ is $zsu(s(r-s))$ (see [7, 10] for a proof) and Lemma 4.3, each term in this sum has virtual degree

$$(k-j)(n+a-k+j) + j(a-1) + (k-j)(k-j-1) + \sum_{i=1}^{t} d_i(j+H_i-D_i+b-1),$$

which is exactly $L_k^{a,b}(B)$. Since the sign alternates, we can only conclude that $A_{n,k}(a,b,q,B)/\prod_{i=1}^{t}[d_i]!$ is symmetric (and not necessarily unimodal), with virtual degree $L_k^{a,b}(B)$. □

7 Unimodality of $A_{n,k}(a,b,q,B)/\prod_{i=1}^{t}[d_i]!$

We will continue to denote the partial sum $h_1+\cdots+h_i$ by H_i, $d_1+\cdots+d_i$ by D_i, and we will also let $E_i = e_1+\cdots+e_i$. We make the convention that $H_0 = D_0 = E_0 = 0$.

Definition 7.1. *Suppose we have integers $h_1,\ldots,h_t$, $d_1,\ldots,d_t$, and $e_1,\ldots,e_t$ with $d_i \in \mathbb{P}$, $h_i \in \mathbb{N}$, and $0 \le e_i \le d_i$. We will denote the vector $(e_1,e_2,\ldots,e_t)$ by $\vec{e}$. For fixed $h_1,\ldots,h_t$ and $d_1,\ldots,d_t$, we define the function*

$$P(\vec{e},x,y) = \prod_{i=1}^{t} \begin{bmatrix} H_i - D_{i-1} + E_{i-1} + y - 1 \\ d_i - e_i \end{bmatrix} \begin{bmatrix} D_i + D_{i-1} - H_i - E_{i-1} + x - y \\ e_i \end{bmatrix}.$$

Lemma 7.2. *Let $B = B(h_1,d_1;\ldots;h_t,d_t)$ be a regular Ferrers board. Then*

$$A_{n,k}(x,y,q,B) =$$

$$\prod_{i=1}^{t}[d_i]! \sum_{\substack{e_1+\cdots+e_t=k,, \\ 0\le e_i\le d_i}} P(\vec{e},x,y) \prod_{i=1}^{t} q^{e_i(H_i-D_i+E_i+y-1)}. \tag{2}$$

Proof. By induction on t, using Corollary 5.10 from [8]. □

Lemma 7.3. *Let $B = B(h_1,d_1;\ldots;h_t,d_t)$ be a regular Ferrers board, $a,b \in \mathbb{N}$ with $a \ge b \ge 1$. Assume that B is such that $d_{i-1}+d_i \ge h_i$ for $1 \le i \le t$ (where $d_0 := 0$). If any of the numerators of the q-binomial coefficients in*

$$P(\vec{e},a,b) = \prod_{i=1}^{t} \begin{bmatrix} H_i - D_{i-1} + E_{i-1} + b - 1 \\ d_i - e_i \end{bmatrix} \begin{bmatrix} D_i + D_{i-1} - H_i - E_{i-1} + a - b \\ e_i \end{bmatrix}$$

are negative, then $P(\vec{e},a,b) = 0$.

Theorem 7.4. *Let $B = B(h_1,d_1;\ldots;h_t,d_t) \subseteq SQ_n$ be a regular Ferrers board such that $d_{i-1}+d_i \ge h_i$ for $1 \le i \le t$. Let $a,b \in \mathbb{N}$ with $a \ge b \ge 1$, and set*

$$L_k^{a,b}(B) = Area(B) + n(b-1) + k(n+a-1) - \sum_{i=1}^{t} d_i D_i$$

as before. Then $A_{n,k}(a,b,q,B)/\prod_{i=1}^{t}[d_i]!$ is $zsu(L_k^{a,b}(B))$.

Proof. We apply Lemma 7.2, which says that

$$\frac{A_{n,k}(a,b,q,B)}{\prod_{i=1}^{t}[d_i]!} = \sum_{e_1+\cdots+e_t=k,\ 0\le e_i\le d_i} P(\vec{e},a,b) \prod_{i=1}^{t} q^{e_i(H_i-D_i+E_i+b-1)},$$

and all of the terms on the right hand side above are in $\mathbb{N}[q]$ by Lemma 7.3. Each term is

$$zsu(\sum_{i=1}^{t}\{(d_i-e_i)(H_i-D_i+E_i+b-1)+e_i(D_i+D_{i-1}-H_i-E_i+a-b)+2e_i(H_i-D_i+E_i+b-1)\}),$$

which a simple calculation shows is the same $zsu(L_k^{a,b}(B))$. Thus by Lemma 4.2, $A_{n,k}(a,b,q,B)/\prod_{i=1}^{t}[d_i]!$ is $zsu(L_k^{a,b}(B))$ as well. □

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