Rook Theory and Cycle-Counting Permutation Statistics

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Abstract

A statistic is found to combinatorially generate the cycle-counting q-hit numbers, defined algebraically by Haglund (*Adv. in Appl. Math.* 17:408-459, 1996). We then define the notion of a cycle-Mahonian pair of statistics (generalizing that of a Mahonian statistic), and show that our newly discovered statistic is part of such a pair. Finally, we note a second example of a cycle-Mahonian pair of statistics which leads us to define the stronger property of being a cycle-Euler-Mahonian pair.

Key Words: rook theory, q-analog, cycle-counting, permutation statistics, major index, Mahonian statistic, Euler-Mahonian statistic.

1 Introduction

In classical rook theory, a *board* is a subset of the $n \times n$ square board (which we shall call SQ_n) depicted in Figure 1. Let $B(b_1, \ldots, b_n)$ denote the board $B \subseteq SQ_n$ consisting of all squares $\{(i,j) \mid j \leq b_i\}$. For example, $B(2,1,3)$ is pictured in Figure 2. When we also have $b_1 \leq b_2 \leq \cdots \leq b_{n-1} \leq b_n$, we call $B(b_1, \ldots, b_n)$ a *Ferrers board*. Another way to specify a Ferrers board, which we will use frequently here, is to give the step heights and depths. The Ferrers board $B(h_1, d_1; \ldots; h_t, d_t)$ is shown in Figure 3. A q-analogue of rook theory, first introduced in [5], focuses on Ferrers boards. In this paper we will concentrate on *regular Ferrers boards*, which are Ferrers boards with the additional property that $b_i \geq i$ for $1 \leq i \leq n$ (or equivalently, $h_1 \geq d_1$, $h_1 + h_2 \geq d_1 + d_2$, $\ldots$, $h_1 + h_2 + \cdots + h_t \geq d_1 + d_2 + \cdots + d_t$ as defined in [7]).

A *rook placement* on a board $B \subseteq SQ_n$ is a subset of squares of B such that no two of these squares lie in the same row or the same column. As the name suggests, these squares represent positions on an $n \times n$ chess board where non-attacking rooks can be placed. We denote the set of all placements of k non-attacking rooks on B by $\mathcal{R}_k(B)$, and the number of ways of placing k non-attacking rooks on B by $r_k(B)$, called the *kth rook number* of B. Note that $r_k(B) = |\mathcal{R}_k(B)|$. The set of all placements of n non-attacking rooks on SQ_n such that exactly k of the rooks lie on B is denoted $\mathcal{H}_{n,k}(B)$. The number of such placements (that is, $|\mathcal{H}_{n,k}(B)|$), written $h_{n,k}(B)$, is called the *kth hit number* of B relative to SQ_n.

Given a placement P of rooks on a Ferrers board $B \subseteq SQ_n$ we can define the following

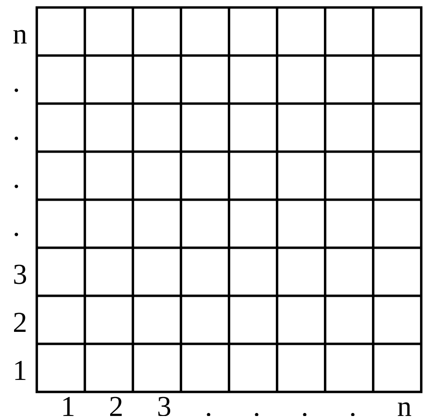

Figure 1: The $n \times n$ square board SQ_n.

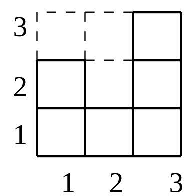

Figure 2: The board $B(2, 1, 3) \subseteq SQ_3$.

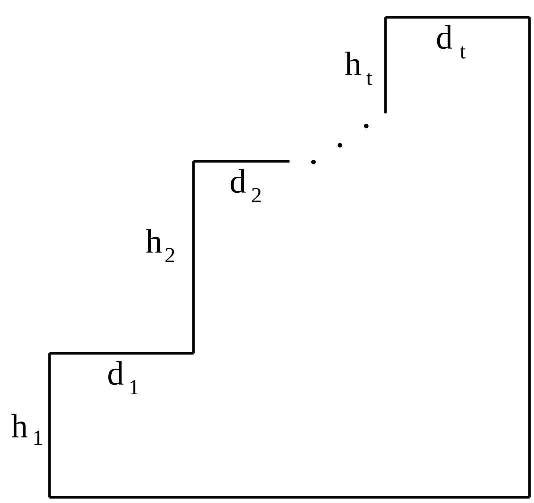

Figure 3: The Ferrers board $B(h_1, d_1; \ldots; h_t, d_t)$.

three statistics for P. First, if we let each rook cancel all squares to the right in its row and below in its column, then as in [5] we can define $inv(P)$ to be the number of uncancelled squares of B. That is, $inv(P)$ is the number of squares on B which are not cancelled by the above scheme and also do not contain a rook from P.

Next, it is possible to associate to a rook placement P on a board $B \subseteq SQ_n$ a simple directed graph G_P on n vertices, a fact first noted in [6] (see also [1] and [2]). A rook from P occupies the square (i, j) if and only if there is an edge from i to j in G_P. We see that G_P is a directed graph on n vertices with some cycles and some directed paths (where vertices with no incident edges count as a directed path of length one). Hence we can define $cyc(P)$ to be the number of cycles in G_P. Note that the definitions of G_P and $cyc(P)$ make sense even if B is not a Ferrers board.

The final statistic depends on the following fact. Let P be any placement of j non-attacking rooks in columns 1 through $i-1$ of a Ferrers board $B = B(b_1, \ldots, b_n)$ (where $j \leq i-1$), and let G_P be the associated directed graph as above. If $b_i \geq i$ then there is exactly one square on B in column i such that placing a rook on this square will complete a new cycle in G_P, whereas if $b_i < i$ then there is no such square on B. This fact can be seen by the following argument.

If $b_i \geq i$, then either there is a directed path in G_P which ends with i or there is not. If there is such a directed path then it must begin with some $k < i$, and (i, k) is the unique square in column i on which placing a rook will complete a cycle in G_P. The square (i, k) lies on B because $k < i \leq b_i$. If there is no such directed path, then placing a rook on (i, i) will complete a cycle in G_P. The square (i, i) clearly lies on B because $b_i \geq i$. Thus we see in this case there is always a unique square on B in column i which will complete a cycle.

If $b_i < i$ and we place a rook on B in column i on square (i,k), we know that $k \leq b_i < i$. In order for the placement of a rook on (i,k) to complete a cycle in G_P, we need a directed path in G_P beginning with k and ending with i. In particular, we must have a rook on the square (ℓ, i) for some $\ell < i$. However, the square (ℓ, i) cannot possibly lie on B because B is a Ferrers board, and hence $\ell < i$ implies that $b_\ell \leq b_i < i$. Thus in this case there is no square in column i of the Ferrers board B which will complete a cycle.

Now suppose P is a placement of some number of rooks on the Ferrers board $B = B(b_1, \ldots, b_n)$. We can then define, for those i with $b_i \geq i$, $s_i(P)$ to be the unique square which, considering only the rooks from P in columns 1 through $i-1$ of B, completes a cycle. Then let $E(P)$ be the number of i such that $b_i \geq i$ and there is no rook from P in column i on or above square $s_i(P)$.

Garsia and Remmel in [5] used the statistic *inv* to define the *kth q-rook number* of a Ferrers board $B = B(b_1, \ldots, b_n) \subseteq SQ_n$ by

$$R_k(q,B) = \sum_{P \in \mathcal{R}_k(B)} q^{inv(P)}$$

and the *q-hit numbers* via the equation

$$\sum_{k=0}^{n} A_{n,k}(q,B) z^k = \sum_{k=0}^{n} R_{n-k}(q,B)[k]!\ z^k \prod_{i=k+1}^{n} (1 - zq^i),$$

where $[n] = 1 + q + q^2 + \cdots + q^{n-1}$ and $[n]! = [n][n-1]\cdots[2][1]$ for $n \in \mathbb{N}$.

Note that both Dworkin [3] and Haglund [8] gave descriptions of different statistics

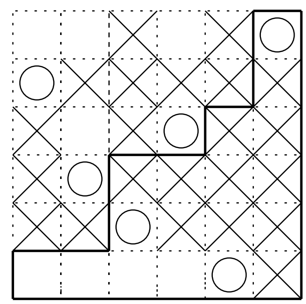

Figure 4: A placement P on $B = B(1, 1, 3, 3, 4, 6) \subseteq SQ_6$ with $s_{B,h}(P) = 8$.

such that

$$A_{n,k}(q, B) = \sum_{P \in \mathcal{H}_{n,n-k}(B)} q^{stat(P)}.$$

Haglund's statistic, which we will denote $s_{B,h}(P)$, is given by the number of squares on SQ_n which neither contain a rook from P nor are cancelled, after applying the following cancellation scheme.

1. Each rook cancels all squares to the right in its row;

2. each rook on B cancels all squares above it in its column;

3. each rook off B cancels all squares below it but off B in its column.

Thus if $B \subseteq SQ_6$ is enclosed by the solid lines in Figure 4 and P is the placement shown, then $s_{B,h}(P) = 8$.

If we let $[y] = (1-q^y)/(1-q)$ for any real number y (generalizing the previous definition of $[n]$ for $n \in \mathbb{N}$), we can now define the *kth cycle-counting q-rook number* of B via

$$R_k(y, q, B) = \sum_{P \in \mathcal{R}_k(B)} [y]^{cyc(P)} q^{inv(P)+(y-1)E(P)}$$

as in [4], and the *cycle-counting q-hit numbers* via the equation

$$\sum_{k=0}^{n} R_{n-k}(y,q,B)[y][y+1]\cdots[y+k-1]z^k \prod_{i=k+1}^{n}(1-zq^{y+i-1}) =$$

$$\sum_{k=0}^{n} A_{n,k}(y,q,B)z^k.$$

What we refer to as $A_{n,k}(y,q,B)$ is the same as $A_k(x,y,B)$ as defined in [7] for the case $x = y$. Note that the $R_k(y,q,B)$ generalize both the q-rook numbers of Garsia and Remmel [5] and the cycle-counting rook numbers discussed in [1], [2] and [7], and the $A_{n,k}(y,q,B)$ analogously generalize both the q-hit numbers and the cycle-counting hit numbers.

In Section 2 of this paper, we find an expression for the $A_{n,k}(y,q,B)$ in terms of the ordinary q-hit numbers of a specific larger board, when $y \in \mathbb{N}$. In Section 3 we define a mapping which takes a placement on the larger board and maps it to a placement on the original board B. We will exploit Haglund's statistic for combinatorially generating the q-hit numbers to prove several useful lemmas about this mapping. In Section 4 we present the main result of this paper, a statistic which combinatorially generates the $A_{n,k}(y,q,B)$. Finally, in Section 5 we apply this statistic to give some new results on permutation statistics involving cycle-counting.

2 $A_{n,k}(y,q,B)$ when $y \in \mathbb{N}$

If $B = B(h_1, d_1; \ldots; h_t, d_t) = B(b_1, \ldots, b_n)$ is a Ferrers board then we define, for $1 \le p \le t$, the Ferrers board

$$B - h_p - d_p := B(h_1, d_1; \ldots; h_p - 1, d_p - 1; \ldots; h_t, d_t).$$

Also, let us denote the number of squares of B by $Area(B)$, so $Area(B) = b_1 + \cdots + b_n$. Finally we define, for $m \in \mathbb{N}$,

$$B_m = B(h_1 + m - 1, d_1; \ldots; h_t, d_t + m - 1).$$

If B is a regular Ferrers board (and hence $b_n = n$), then B_m is regular with $n+m-1$ columns, of heights $b_1 + m - 1,\ b_2 + m - 1,\ \ldots,\ b_n + m - 1,\ \underbrace{n+m-1,\ \ldots,\ n+m-1}_{m-1}$. Note since at least the last m columns of $B_m \subseteq SQ_{n+m-1}$ for any regular Ferrers board B have height $n+m-1$, any rooks in the last m columns of SQ_{n+m-1} must be on B_m. Thus in particular any placement of $n+m-1$ rooks on SQ_{m+n-1} must have at least m rooks on B_m, so $\mathcal{H}_{n+m-1,k}(B_m) = \emptyset$ for $0 \le k \le m-1$.

We use the following two lemmas to prove the main proposition of this section.

Lemma 2.1. *For B a regular Ferrers board, $m \in \mathbb{N}$ and B_m as defined above,*

$$A_{n,0}(m,q,B) = A_{n+m-1,0}(q, B_m)/[m-1]!.$$

Proof. By definition $A_{n,0}(y,q,B) = R_n(y,q,B)$, and by (47) of [7] with $x = 0$,

$$R_n(y,q,B) = \prod_{i=1}^{n} [b_i - i + y] = \prod_{i=1}^{n} [(b_i + y - 1) - i + 1].$$

Hence $A_{n,0}(m,q,B) = \prod_{i=1}^{n}[(b_i+m-1)-i+1]$. By the definition of Haglund's statistic for generating the q-hit numbers,

$$A_{n+m-1,0}(q,B_m) = [b_1+m-1][(b_2+m-1)-1]\cdots[(b_n+m-1)-n+1]\times$$

$$[(n+m-1)-n][(n+m-1)-n-1]\cdots[(n+m-1)-n-m+2] =$$

$$\prod_{i=1}^{n}[(b_i+m-1)-i+1]\times[m-1]!,$$

and the lemma follows. □

Lemma 2.2. *For any regular Ferrers board $B = B(h_1,d_1;\ldots;h_t,d_t)$ we have that*

$$A_{n,k}(y,q,B) = [y+k+d_t-1]A_{n-1,k}(y,q,B-h_t-d_t)+$$

$$q^{y+k+d_t-2}[n-k-d_t+1]A_{n-1,k-1}(y,q,B-h_t-d_t)$$

for $0 \leq k \leq n$.

Proof. Let $x = y$ and $p = t$ in Lemma 5.7 of [7]. □

The next proposition is integral to proving the main result of the paper in Section 4.

Proposition 2.3. *For any regular Ferrers board B and $m \in \mathbb{N}$, we have that*

$$A_{n,k}(m,q,B) = \frac{A_{n+m-1,k}(q,B_m)}{[m-1]!}$$

for $0 \leq k \leq n$.

Proof. We will prove this proposition by induction on $Area(B)$. When $Area(B) = 1$ the only regular Ferrers board is the 1×1 square SQ_1, and an easy calculation shows that $A_{1,0}(m,q,SQ_1) = [m]$ and $A_{1,k}(m,q,SQ_1) = 0$ for all $k > 0$. By the definition of $s_{B_m,h}(P)$

given in Section 1, $A_{1+m-1,0}(q, B_m) = A_{m,0}(q, SQ_m) = [m]!$ and $A_{m,k}(q, SQ_m) = 0$ for $k > 0$, so the proposition holds in this case.

Now assume the proposition holds for all regular Ferrers boards of $Area < A$, and suppose $B = B(h_1, d_1; \ldots; h_t, d_t) = B(b_1, \ldots, b_n)$ is such that $Area(B) = A$. By Lemma 2.1, we have that $A_{n,0}(m, q, B) = A_{0,n+m-1}(q, B_m)/[m-1]!$. Then by Lemma 2.2 when $y = m$, we have for $k > 0$ that $A_{n,k}(m, q, B)$ equals

$$[m+k+d_t-1]A_{n-1,k}(m, q, B-h_t-d_t)+q^{m+k+d_t-2}[n-k-d_t+1]A_{n-1,k-1}(m, q, B-h_t-d_t),$$

which is

$$[k + (d_t + m - 1)]A_{n-1,k}(m, q, B - h_t - d_t)+$$
$$q^{k+(d_t+m-1)-1}[(n + m - 1) - (d_t + m - 1) - k + 1]A_{n-1,k-1}(m, q, B - h_t - d_t). \qquad (1)$$

By induction, $A_{n-1,k}(m, q, B - h_t - d_t) = A_{(n-1)+m-1,k}(q, (B - h_t - d_t)_m)/[m - 1]!$, which equals

$$A_{(n-1)+m-1,k}(q, B(h_1 + m - 1, d_1; \ldots; h_t - 1, d_t - 1 + m - 1))/[m - 1]!,$$

and $A_{n-1,k-1}(m, q, B - h_t - d_t)$ is

$$A_{(n-1)+m-1,k-1}(q, B(h_1 + m - 1, d_1; \ldots; h_t - 1, d_t - 1 + m - 1)/[m - 1]!.$$

Thus (1) is equal to

$$[k + (d_t + m - 1)]A_{(n-1)+m-1,k}(q, B(h_1 + m - 1, d_1; \ldots; h_t - 1, d_t - 1 + m - 1))/[m - 1]!$$
$$+q^{k+(d_t+m-1)-1}[(n + m - 1) - (d_t + m - 1) - k + 1]\times$$
$$A_{(n-1)+m-1,k-1}(q, B(h_1 + m - 1, d_1; \ldots; h_t - 1, d_t - 1 + m - 1)/[m - 1]!$$

which is

$$\frac{1}{[m-1]!}\Big\{[k+(d_t+m-1)]A_{(n-1)+m-1,k}(q,B(h_1+m-1,d_1;\ldots;h_t-1,d_t-1+m-1))$$

$$+q^{k+(d_t+m-1)-1}[(n+m-1)-(d_t+m-1)-k+1]\times$$

$$A_{(n-1)+m-1,k-1}(q,B(h_1+m-1,d_1;\ldots;h_t-1,d_t-1+m-1)\Big\}.$$

Now by Lemma 2.2 with $y=1$, the above is equal to

$$\frac{1}{[m-1]!}A_{n+m-1,k}(q,B(h_1+m-1,d_1;\ldots;(h_t-1)+1,(d_t-1+m-1)+1)=$$

$$\frac{1}{[m-1]!}A_{n+m-1,k}(q,B(h_1+m-1,d_1;\ldots;h_t,d_t+m-1)),$$

which is

$$\frac{1}{[m-1]!}A_{n+m-1,k}(q,B_m)$$

and the proposition follows. □

3 The map $\phi_{n,B,m}$ and its properties

For any Ferrers board $F\subseteq SQ_d$, let us denote $\cup_{i=0}^{d}\mathcal{H}_{d,i}(F)$ by $\mathcal{P}_d(F)$. Throughout this section let $B\subseteq SQ_n$ be some fixed regular Ferrers board, $B_m\subseteq SQ_{n+m-1}$ as previously defined for some $m\in\mathbb{N}$. If $P\in\mathcal{P}_{n+m-1}(B_m)$, let $r_i(P)$ denote the rook from P in the ith column of SQ_{n+m-1}, and analogously for $Q\in\mathcal{P}_n(B)$ and $r_i(Q)$.

We define a mapping $\phi_{n,B,m}:\mathcal{P}_{n+m-1}(B_m)\to\mathcal{P}_n(B)$ as follows. Suppose $P\in\mathcal{P}_{n+m-1}(B_m)$. Beginning in column 1 and proceeding from left to right one column at a time, the following occurs.

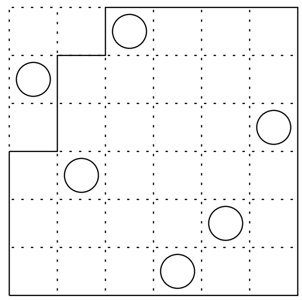

Figure 5: A placement P of rooks on $B_m \subseteq SQ_{n+m-1}$ for $B = B(1,3,4,4)$, $m = 3$, and $n = 4$.

1. $r_i(P)$ is on one of the m lowest squares in column i not attacked by a rook to the left if and only if $r_i(P)$ maps to the unique square $s_i(\phi_{n,B,m}(P))$ which completes a cycle in the image of P so far. (That is, you consider the placement of rooks on $SQ_n \supseteq B$ in columns 1 through $i-1$ given by $\phi_{n,B,m}(r_1(P)), \phi_{n,B,m}(r_2(P)), \ldots, \phi_{n,B,m}(r_{i-1}(P))$, and $s_i(\phi_{n,B,m}(P))$ is the unique square in column i which would complete a cycle in this placement.)

2. Otherwise, $r_i(P)$ is on the $(m+a_i)$th square ($a_i > 0$) in column i not attacked by a rook to the left if and only if $r_i(P)$ maps to the a_ith available square in column i of B so far *which does not complete a cycle* (that is, the a_ith available square in column i of B, not counting the square $s_i(\phi_{n,B,m}(P))$ described above).

The best way to understand this mapping is to do an example in detail. Consider the placement P of 6 rooks on the board $SQ_6 \supseteq B_3$, where $B = B(1,3,4,4) \subseteq SQ_4$. This board and placement are depicted in Figure 5. The leftmost rook $r_1(P)$ is in the fifth available position in its column, which is also the fifth square in this column not attacked

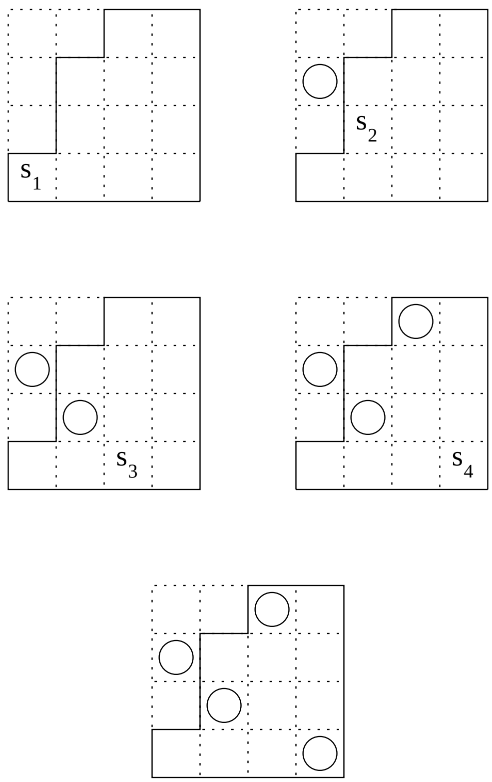

Figure 6: The image of P from Figure 5 under $\phi_{4,B,3}$ at each step; the cycle square in column i is denoted s_i.

by a rook to the left (because there are no rooks to the left). Since $m = 3$ in this case (so $5 = m + 2$), $\phi_{4,B,3}(r_1(P))$ is on the second available square in column 1 of SQ_4 which does not complete a cycle. Since the square $(1, 1)$ is always the cycle square in the first column, $r_1(P)$ maps to square $(1, 3)$.

Now the cycle square in column 2 of B is $(2, 2)$. Since $r_2(P)$ is on one of the 3 lowest squares in column 2 of SQ_6 not attacked by a rook to the left, $\phi_{4,B,3}(r_2(P))$ is on the cycle square $(2, 2)$.

At this point the cycle square is $(3, 1)$. Here $r_3(P)$ is on the fourth square not attacked by a rook to the left (and $4 = m + 1$), so $\phi_{4,B,3}(r_3(P))$ is on the first available square of SQ_4 which does not complete a cycle. In this case square $(3, 1)$ is the cycle square, and squares $(3, 2)$ and $(3, 3)$ are attacked by the rooks in columns 1 and 2 of SQ_4, so the first available non-cycle square is $(3, 4)$.

Finally, the cycle square in column 4 of SQ_4 is $(4, 1)$. Since $r_4(P)$ in on the lowest square in its column (and hence one of the 3 lowest not attacked by a rook to the left), $\phi_{4,B,3}(r_4(P))$ is on the cycle square. The image $\phi_{4,B,3}(P)$ is depicted in Figure 6.

The general principle behind $\phi_{n,B,m}$ is the following. Suppose you want to map a rook in column i of a placement P on $SQ_{n+m-1} \supseteq B_m$. Imagine covering columns $i+1$ through $n + m - 1$ of SQ_{n+m-1}, so that only columns 1 through i can be seen. If $r_i(P)$ is on one of the m lowest available squares in column i of this "covered" board, then $r_i(P)$ maps to the square of $SQ_n \supseteq B$ which completes a cycle in the image so far. The remaining $(n + m - 1) - (i - 1) - m = n - i$ squares in column i of SQ_{n+m-1} are then mapped in order to the $n - (i - 1) - 1 = n - i$ available non-cycle squares in column i of SQ_n.

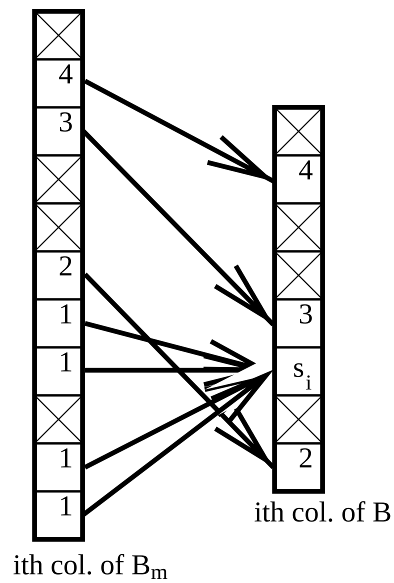

Figure 7: The general idea behind the map$\phi_{n,B,m}$ in the ith column.

Figure 7 illustrates this idea further.

Note that in the above definition of $\phi_{n,B,m}$ we ignore the rooks from a placement $P \in \mathcal{P}_{n+m-1}(B_m)$ in columns $n+1$ through $n+m-1$ of SQ_{n+m-1}. Thus for a fixed arrangement of n rooks in columns 1 through n of SQ_{n+m-1}, we see there will be $(m-1)!$ total ways to arrange the rooks in the last $m-1$ columns of SQ_{n+m-1}. Hence these $(m-1)!$ placements will all map to the same placement of n rooks on SQ_n.

We have the following lemmas.

Lemma 3.1. *$\phi_{n,B,m}$ is surjective.*

Proof. Given a placement $Q \in \mathcal{P}_n(B)$, we build a placement $P \in \mathcal{P}_{n+m-1}(B)$ from left to right. If the rook from Q in the ith column is on the square which completes a cycle, then we choose $r_i(P)$ to be on one of the m lowest available squares of SQ_{n+m-1} (so for each rook on a cycle square from Q, we will have m choices for the rook from P in the same column). If $r_i(Q)$ is on the a_ith square in its column not attacked by a rook to the left and which does not complete a cycle, then $r_i(P)$ must be on the $(m + a_i)$th available square in column i of SQ_{n+m-1}. Once the rooks in columns 1 through n are determined, we choose any arrangement of rooks in columns $n + 1$ through $n + m - 1$ which results in a non-attacking placement. It is clear that this procedure will result in a placement $P \in \mathcal{P}_{n+m-1}(B)$, and each rook from P was chosen to ensure that $Q = \phi_{n,B,m}(P)$. □

Lemma 3.2. *Let $B \subseteq SQ_n$ be a regular Ferrers board, $m \in \mathbb{N}$, $\phi_{n,B,m} : \mathcal{P}_{n+m-1}(B_m) \to \mathcal{P}_n(B)$. Let $P \in \mathcal{P}_{n+m-1}(B_m)$, and $Q = \phi_{n,B,m}(P)$. For $1 \leq i \leq n$, $r_i(P)$ is on B_m if and only if $r_i(Q)$ is on B, and $r_i(P)$ is off B_m on square $(i, j_i + m - 1)$ if and only if $r_i(Q)$ is off B on square (i, j_i).*

Proof. Fix n, B and m; the proof is by induction on i. If $i = 1$, then by definition of $\phi_{n,B,m}$ any rook on one of the m lowest squares in column 1 maps to the unique square in column 1 of B which completes a cycle, namely $(1, 1)$, and a rook on square $(1, j + m - 1)$ (for $j > 1$) maps to square $(1, j)$. Thus $r_1(P)$ is on B_m if and only if $r_1(Q)$ is on B, and $r_1(P)$ is off B_m on square $(1, j + m - 1)$ if and only if $r_1(Q)$ is off B on square $(1, j)$ as desired.

Now consider the rook $r_i(P)$ in column i of P for $i > 1$. Let k_i denote the number of rooks from P in columns 1 through $i - 1$ which can attack a square on B_m in column i; that is, k_i is the number of rooks in columns 1 through $i - 1$ of SQ_{n+m-1} which are in rows 1 through $b_i + m - 1$, where b_i denotes the height of column i of B. Then we see that there are $b_i + m - 1 - k_i$ available squares in column i of SQ_{n+m-1} which are on B_m.

By induction, any rook from P in columns 1 through $i-1$ is on B_m if and only if this rook maps to a rook on B, and any rook is off B_m on square $(s, j_s + m - 1)$ if and only if this rook maps to a rook off B on square (s, j_s). These two facts imply that a rook in columns 1 through $i-1$ in a row between 1 and $b_i + m - 1$ of SQ_{n+m-1} maps to a rook in columns 1 through $i-1$ of SQ_n in a row between 1 and b_i. Thus the number of rooks in columns 1 through $i-1$ of SQ_n from Q which can attack a square on B is also k_i, and hence there are $b_i - k_i$ available squares in column i of SQ_n which are on B.

A rook on one of the lowest m available squares in column i of SQ_{n+m-1} will map to the unique square in column i of SQ_n which completes a cycle in Q. Since B is a regular Ferrers board, this square will lie on B. Thus there are $(b_i + m - 1) - k_i - m = b_i - k_i - 1$ available squares on B_m in column i of SQ_{n+m-1} which do not map to $s_i(Q)$. On SQ_n we see that there is one square which completes a cycle in Q, and $b_i - k_i - 1$ squares which do not complete a cycle. Hence by the definition of $\phi_{n,B,m}$ the $b_i - k_i - 1$ squares on B_m which do not map to $s_i(Q)$ are in one to one correspondence with the $b_i - k_i - 1$ available squares on B in column i, so $r_i(P)$ is on B_m if and only if $r_i(Q)$ is on B.

Finally, the remaining $(n + m - 1) - (b_i + m - 1) - (i - 1 - k_i) = n - b_i - i + 1 + k_i$ available squares in column i of SQ_{n+m-1} off B_m are in one-to-one correspondence with the $n - b_i - (i - 1 - k_i) = n - b_i - i + 1 + k_i$ available squares in column i of SQ_n off B. By induction a rook on square $(s, j_s + m - 1)$ for $1 \leq s \leq i - 1$ which is off B_m maps to a rook on square (s, j_s) which is off B. Thus we see that in column i a square $(i, j_i + m - 1)$ off B_m is available if and only if the square (i, j_i) (which is off B) is available. Thus by definition of $\phi_{n,B,m}$ we see that $r_i(P)$ is off B_m on square (i, j_i+m-1) if and only if $r_i(Q)$ is off B on square (i, j_i). □

Note that a corollary of Lemma 3.2 is that $\phi_{n,B,m}|_{\mathcal{H}_{n+m-1,(n+m-1)-k}(B_m)}$ is actually a map from $\mathcal{H}_{n+m-1,(n+m-1)-k}(B_m)$ to $\mathcal{H}_{n,n-k}(B)$.

Now let us weight a placement $Q \in \mathcal{H}_{n,k}(B)$ by

$$\sum_{P \in \phi^{-1}_{n,B,m}(Q)} q^{s_{B_m,h}(P)}, \tag{2}$$

where $s_{B_m,h}(P)$ is as described in Section 1. As was earlier discussed, the rooks from some $P \in \mathcal{P}_{n+m-1}(B_m)$ in columns $n+1$ through $n+m-1$ will all lie on B_m. Thus by the definition of Haglund's statistic $s_{B_m,h}(P)$, if we fix the rooks in the first n columns and sum over all the possible $(m-1)!$ placements of non-attacking rooks in the last $m-1$ columns, we will generate $[m-1]!$.

Given a statistic *stat* which can be calculated for any rook placement R on a board $SQ_d \supseteq F$, we will denote by $stat(R)_i$ the contribution to $stat(R)$ coming from the ith column of SQ_d. Thus for $Q \in \mathcal{H}_{n,k}(B)$, we see that

$$\sum_{P \in \phi^{-1}_{n,B,m}(Q)} q^{s_{B_m,h}(P)} = [m-1]! \sum_{P'} \prod_{i=1}^{n} q^{s_{B_m,h}(P)_i}$$

where the second sum is over all placements P' of rooks in columns 1 through n of $SQ_{n+m-1} \supseteq B_m$ which extend to a placement $P \in \phi^{-1}_{n,B,m}(Q)$ and P is any one of these extensions of P'.

We have the following lemmas about this weighting.

Lemma 3.3. *For a fixed placement $Q \in \mathcal{H}_{n,k}(B)$, suppose a rook $r_i(Q)$ is on the square $s_i(Q)$. Then*

$$\sum_{P \in \phi^{-1}_{n,B,m}(Q)} q^{s_{B_m,h}(P)_i} = [m].$$

Proof. If $r_i(Q)$ is on $s_i(Q)$, then by definition for $P \in \phi^{-1}_{n,B,m}(Q)$ $r_i(P)$ is on one of the m lowest squares in column i not attacked by a rook to the left. The lowest square gives a contribution from column i of 1, the second lowest a contribution of q, …, the mth lowest a contribution of q^{m-1}. Thus we see that $\sum_{P \in \phi^{-1}_{n,B,m}(Q)} q^{s_{B_m,h}(P)_i} = [m]$. □

Lemma 3.4. *For a fixed placement $Q \in \mathcal{H}_{n,k}(B)$, suppose a rook $r_i(Q)$ is below the square $s_i(Q)$ on the a_ith square not attacked by a rook to the left. Then for every $P \in \phi^{-1}_{n,B,m}(Q)$, $r_i(P)$ contributes a factor of q^{m-1+a_i} to each summand of (2).*

Proof. $r_i(Q)$ is on the a_ith square not attacked by a rook to the left, which is also (since $r_i(Q)$ is below $s_i(Q)$) the a_ith square not attacked by a rook to the left which does not complete a cycle. Thus we see by the definition of $\phi_{n,B,m}$ that $r_i(P)$ must be on the $(m+a_i)$th square in column i of SQ_{n+m-1} not attacked by a rook to the left. Since $r_i(Q)$ is below $s_i(Q)$ it must be on B, so by Lemma 3.2 $r_i(P)$ is on B_m. Thus $r_i(P)$ has $m-1+a_i$ uncancelled squares below it, so it contributes $m-1+a_i$ to $s_{B_m,h}(P)$ and hence a factor of q^{m-1+a_i} to each summand of (2). □

Lemma 3.5. *For a fixed placement $Q \in \mathcal{H}_{n,k}(B)$, suppose a rook $r_i(Q)$ on B is above the square $s_i(Q)$, and on the a_ith square not attacked by a rook to the left. Then for every $P \in \phi^{-1}_{n,B,m}(Q)$, $r_i(P)$ contributes a factor of q^{m-1+a_i-1} to each summand of (2).*

Proof. $r_i(Q)$ is on the a_ith square not attacked by a rook to the left, which is (since $r_i(Q)$ is above $s_i(Q)$) the $(a_i - 1)$th square not attacked by a rook to the left which does not complete a cycle. Thus we see by the definition of $\phi_{n,B,m}$ that $r_i(P)$ must be on the $(m + a_i - 1)$th square in column i of SQ_{n+m-1} not attacked by a rook to the left. Again by Lemma 3.2, since $r_i(Q)$ is on B $r_i(P)$ must be on B_m. Thus $r_i(P)$ has $m - 1 + a_i - 1$ uncancelled squares below it, so it contributes $m - 1 + a_i - 1$ to $s_{B_m,h}(P)$ and hence a factor of q^{m-1+a_i-1} to each summand of (2). ☐

Lemma 3.6. *For a fixed placement $Q \in \mathcal{H}_{n,k}(B)$, suppose a rook $r_i(Q)$ is off B. Then for every $P \in \phi^{-1}_{n,B,m}(Q)$, $r_i(P)$ contributes a factor of $q^{m-1+s_{B,h}(Q)_i}$ to each summand of (2).*

Proof. By Lemma 3.2 and its proof, we see that if $r_i(Q)$ is on (i, j) then $r_i(P)$ is on $(i, j + m - 1)$ and the number of rooks below and to the left of $r_i(Q)$ is equal to the number of rooks below and to the left of $r_i(P)$. Thus the number of squares coming from column i when calculating $s_{B_m,h}(P)$ is the same as the number of squares from column i when calculating $m - 1 + s_{B,h}(Q)$, hence such a rook contributes a factor of $q^{m-1+s_{B,h}(Q)_i}$ to each summand of (2). ☐

Note that for $Q \in \mathcal{P}_n(B)$ and $r_i(Q)$ not on the cycle square but on the a_ith square not attacked by a rook to the left, $a_i = s_{B,h}(Q)_i + 1$. Thus for a rook below the cycle square in column i we have that $q^{m-1+a_i} = q^{m-1+s_{B,h}(Q)_i+1}$, and for a rook on B above the cycle square in column i, $q^{m-1+a_i-1} = q^{m-1+s_{B,h}(Q)_i}$. Now we see that

$$A_{n,k}(m, q, B) = \frac{1}{[m-1]!} A_{n+m-1,k}(q, B_m) =$$

$$\frac{1}{[m-1]!}\sum_{P\in\mathcal{H}_{n+m-1,(n+m-1)-k}(B_m)} q^{s_{B_m,h}(P)} = \frac{1}{[m-1]!}\sum_{Q\in\mathcal{H}_{n,n-k}(B)}\left\{\sum_{P\in\phi^{-1}_{n,B,m}(Q)} q^{s_{B_m,h}(P)}\right\} =$$
$$\frac{1}{[m-1]!}[m-1]!\sum_{Q\in\mathcal{H}_{n,n-k}(B)}\left\{\sum_{P' \text{ which extend to some } P\in\phi^{-1}_{n,B,m}(Q)}\prod_{i=1}^{n} q^{s_{B_m,h}(P)_i}\right\} =$$
$$\sum_{Q\in\mathcal{H}_{n,n-k}(B)}[m]^{cyc(Q)}\prod_{r_i(Q) \text{ below } s_i(Q)} q^{m-1+a_i(Q)}\times$$
$$\prod_{r_i(Q) \text{ above } s_i(Q) \text{ on } B} q^{m-1+a_i(Q)-1}\prod_{r_i(Q) \text{ above } s_i(Q) \text{ off } B} q^{m-1+s_{B,h}(Q)_i} =$$
$$\sum_{Q\in\mathcal{H}_{n,n-k}(B)}[m]^{cyc(Q)}\prod_{r_i(Q) \text{ below } s_i(Q)} q^{(m-1)+s_{B,h}(Q)_i+1}\prod_{r_i(Q) \text{ above } s_i(Q)} q^{(m-1)+s_{B,h}(Q)_i} =$$
$$\sum_{Q\in\mathcal{H}_{n,n-k}(B)}[m]^{cyc(Q)}q^{(n-cyc(Q))(m-1)+s_{B,b}(Q)+E(Q)}, \qquad (3)$$

where $s_{B,b}(Q)$ is defined as the number of squares on SQ_n which neither contain a rook from P nor are cancelled, after applying the following cancellation scheme.

1. Each rook cancels all squares to the right in its row;
2. each rook on B cancels all squares above it in its column (squares both on B and strictly above B);
3. each rook on B which also completes a cycle cancels all squares below it in its column as well;
4. each rook off B cancels all squares below it but above B.

Note that if we let $m = 1$ in (3), then we obtain a statistic to generate the q-hit numbers. That is,

$$A_{n,k}(q,B) = \sum_{Q\in\mathcal{H}_{n,n-k}(B)} q^{s_{B,b}(Q)+E(Q)}.$$

While this new statistic is equal to neither that of Haglund [8] nor Dworkin [3], it is a member of the family of statistics discussed by Haglund and Remmel [9, p. 39].

4 The main theorem and a corollary

We can now define

$$\tilde{A}_{n,k}(y,q,B) = \sum_{P \in \mathcal{H}_{n,n-k}(B)} [y]^{cyc(P)} q^{(n-cyc(P))(y-1)+s_{B,b}(P)+E(P)}$$

and prove the following.

Theorem 4.1. *For B any regular Ferrers board we have*

$$A_{n,k}(y,q,B) = \tilde{A}_{n,k}(y,q,B)$$

for $0 \leq k \leq n$.

Proof. Both of the above expressions are polynomials in the variable q^y over the field $\mathbb{Q}(q)$ of fixed degree. By the previous section, $A_{n,k}(m,q,B) = \tilde{A}_{n,k}(m,q,B)$ for any $m \in \mathbb{N}$. Thus these two polynomials have infinitely many common values, hence must be equal for all y. □

A permutation statistic s is called *Mahonian* if

$$\sum_{\sigma \in S_n} q^{s(\sigma)} = [n]!.$$

We shall say that a pair (s_1, s_2) of statistics is *cycle-Mahonian* if

$$\sum_{\sigma \in S_n} [y]^{s_1(\sigma)} q^{s_2(\sigma,y)} = [y][y+1]\cdots[y+n-1].$$

Note that the statistic s_2 may depend on both σ and y. This notion generalizes that of a Mahonian statistic, since letting $y = 1$ in the definition of cycle-Mahonian gives

$$\sum_{\sigma \in S_n} q^{s_2(\sigma,1)} = [1][2]\cdots[n] = [n]!.$$

We can associate to a permutation $\sigma \in S_n$ the placement P_σ of n rooks on SQ_n consisting of the squares $\{(i,j) \mid \sigma(i) = j\}$. We can then make any statistic *stat* defined for placements of n rooks on SQ_n into a permutation statistic by letting

$$stat(\sigma) = stat(P_\sigma).$$

In light of this definition, we have the following.

Corollary 4.2. *The pair* $(cyc(-), (n-cyc(-))(y-1)+s_{B,b}(-)+E(-))$ *is cycle-Mahonian for any regular Ferrers board* $B \subseteq SQ_n$.

Proof. By definition,

$$\sum_{\sigma \in S_n} [y]^{cyc(\sigma)} q^{(n-cyc(\sigma))(y-1)+s_{B,b}(\sigma)+E(\sigma)} = \sum_{\sigma \in S_n} [y]^{cyc(P_\sigma)} q^{(n-cyc(P_\sigma))(y-1)+s_{B,b}(P_\sigma)+E(P_\sigma)}.$$

By Theorem 4.1 we know that

$$\sum_{\sigma \in S_n} [y]^{cyc(P_\sigma)} q^{(n-cyc(P_\sigma))(y-1)+s_{B,b}(P_\sigma)+E(P_\sigma)} = \sum_{k=0}^{n} A_{n,k}(y,q,B).$$

Finally, it is known [7] that for any regular Ferrers board $B \subseteq SQ_n$,

$$\sum_{k=0}^{n} A_{n,k}(y,q,B) = [y][y+1]\cdots[y+n-1]. \tag{4}$$

□

5 A cycle-Euler-Mahonian pair

Recall the permutation statistics for $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$

$$des(\sigma) = \left|\{i \mid \sigma_i > \sigma_{i+1}\}\right| \quad \text{and} \quad maj(\sigma) = \sum_{\sigma_i > \sigma_{i+1}} i,$$

called the *number of descents* and the *major index*, respectively, of the permutation σ. The *q-Eulerian numbers* are then defined by the equation

$$E_{n,k}(q) = \sum_{\sigma \in S_n,\ des(\sigma)=k-1} q^{maj(\sigma)}.$$

It is known [8] that $E_{n,k}(q) = A_{n,k-1}(q, \mathbb{T}_n)$, where $\mathbb{T}_n = B(1, 2, \ldots n)$ is the triangular board. Hence we obtain a q, y-version of the Eulerian numbers via the equation

$$E_{n,k}(y, q) = A_{n,k-1}(y, q, \mathbb{T}_n).$$

Now suppose $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$. If $\sigma_{j_1} = 1$, let y_1 be the cycle $(\sigma_1 \cdots \sigma_{j_1})$. If α is the smallest integer not contained in y_1, and $\sigma_{j_2} = \alpha$, let y_2 be the cycle $(\sigma_{j_1+1} \cdots \sigma_{j_2})$, etc. If the result of the above procedure is the product $y_1 y_2 \cdots y_p$, we will let $p = \ell rmin(\sigma)$, called the number of *left-to-right minima* of σ. We can now define

$$\tilde{E}_{n,k}(y, q) = \sum_{\sigma \in S_n,\ des(\sigma)=k-1} [y]^{\ell rmin(\sigma)} q^{(n-\ell rmin(\sigma))(y-1)+maj(\sigma)}$$

and prove the following.

Proposition 5.1. *We have*

$$\tilde{E}_{n,k}(y, q) = [y + k - 1]\tilde{E}_{n-1,k}(y, q) + q^{y+k-2}[n - k + 1]\tilde{E}_{n-1,k-1}(y, q)$$

for any $n, k \in \mathbb{N}$.

Proof. We mimic the well known proof when $y = 1$ (that is, in the case of the regular q-Eulerian numbers $E_{n,k}(q)$). Any permutation in S_n with $k - 1$ descents can be built from one in S_{n-1} with either $k - 1$ or $k - 2$ descents in the following way.

First suppose $\sigma \in S_{n-1}$ has $k-1$ descents, occurring at positions $i_1, i_2, \ldots, i_{k-1}$. Thus $\sigma = \sigma_1 \cdots \sigma_{i_1} \cdots \sigma_{i_{k-1}} \cdots \sigma_{n-1}$, where

$$\sigma_1 < \sigma_2 < \cdots < \sigma_{i_1} > \sigma_{i_1+1} < \cdots < \sigma_{i_2} > \sigma_{i_2+1} < \cdots < \sigma_{i_{k-1}} > \sigma_{i_{k-1}+1} < \cdots < \sigma_{n-1}.$$

This permutation will contribute $[y]^{\ell rmin(\sigma)} q^{((n-1)-\ell rmin(\sigma))(y-1)+maj(\sigma)}$ to $\tilde{E}_{n-1,k}(y,q)$.

We can place n in any of the $k-1$ positions of σ where a descent occurs, thereby creating a new permutation σ' in S_n which still has only $k-1$ descents. If we place n in the (i_1+1)th position, all the descents are moved one position to the right, thus increasing maj by $k-1$. Here we see that $\ell rmin(\sigma) = \ell rmin(\sigma')$, since there will clearly be a number to the right of where we have placed n which is smaller than n. However, we have increased the number of letters in the permutation from $n-1$ to n. Thus

$$[y]^{\ell rmin(\sigma')} q^{(n-\ell rmin(\sigma'))(y-1)+maj(\sigma')} = \{q^{(y-1)+(k-1)}\} \times [y]^{\ell rmin(\sigma)} q^{((n-1)-\ell rmin(\sigma))(y-1)+maj(\sigma)}.$$

Next we see that if we place n in the (i_2+1)th position, this time maj will increase by $k-2$, and again $\ell rmin(\sigma') = \ell rmin(\sigma)$ but the number of letters in the permutation increases by one. Therefore in this case, we gain a factor of $q^{(y-1)+(k-2)}$.

Continuing in this manner we proceed from left to right. Placing n in the $(i_{k-1}+1)$th position gives a factor of $q^{(y-1)+1}$, so the sum of all of these factors is $q^{y+k-2} + q^{y+k-3} + \cdots + q^{y+1} + q^y$. There is one last position where we can place n and not increase des, and that is the nth position. This will also not increase maj, however $\ell rmin(\sigma')$ will now be $\ell rmin(\sigma)+1$. We have also increased the total number of letters from $n-1$ to n, but since $\ell rmin(\sigma') = \ell rmin(\sigma) + 1$ we have that $(n-1) - \ell rmin(\sigma) = n - \ell rmin(\sigma')$. Thus this last placement of n just contributes $[y]$, and summing over all positions for n which do not increase $des(\sigma)$ gives $[y]+q^y+q^{y+1}+\cdots+q^{y+k-2}$, which is equal to $[y+k-1]$. Summing again, over all $\sigma \in S_{n-1}$ with $k-1$ descents yields the first term in the recurrence.

Now suppose $\sigma \in S_{n-1}$ has $k-2$ descents, occurring at positions $i_1, i_2, \ldots, i_{k-2}$. Thus $\sigma = \sigma_1 \cdots \sigma_{i_1} \cdots \sigma_{i_{k-2}} \cdots \sigma_{n-1}$, where

$$\sigma_1 < \sigma_2 < \cdots < \sigma_{i_1} > \sigma_{i_1+1} < \cdots < \sigma_{i_2} > \sigma_{i_2+1} < \cdots < \sigma_{i_{k-2}} > \sigma_{i_{k-2}+1} < \cdots < \sigma_{n-1}.$$

This permutation will contribute $[y]^{\ell rmin(\sigma)} q^{((n-1)-\ell rmin(\sigma))(y-1)+maj(\sigma)}$ to $\tilde{E}_{n-1,k-1}(y,q)$.

We can place n in any of the $n-(k-1)$ positions which will create an additional descent in our new permutation σ'. If we place n in the first position, this new descent will add 1 to maj, and it will move each of the $k-2$ descents to the right of it one position to the right, adding another $k-2$ to maj. Thus maj will increase by a total of $k-1$. As argued in the above case, $\ell rmin(\sigma') = \ell rmin(\sigma)$, but since we have increased the number of letters in the permutation from $n-1$ to n, $n - \ell rmin(\sigma') = \{(n-1) - \ell rmin(\sigma)\} + 1$. Thus we also obtain an extra q^{y-1}, and hence

$$[y]^{\ell rmin(\sigma')} q^{(n-\ell rmin(\sigma'))(y-1)+maj(\sigma')} = \{q^{(y-1)+(k-1)}\} \times [y]^{\ell rmin(\sigma)} q^{((n-1)-\ell rmin(\sigma))(y-1)+maj(\sigma)}.$$

Continuing in this manner until the first descent at position i_1, we obtain factors of $q^{(y-1)+(k-1)}, q^{(y-1)+k}, \ldots, q^{(y-1)+k-2+i_1}$. We do not place n in the (i_1+1)th position, as this will not create a new descent. Instead, we skip over this position and move to the (i_1+2)th position. The new descent created will contribute i_1+2 to maj. Now there will be only $k-3$ descents to the right of where we have placed n, which will each be moved one position to the right increasing maj by $k-3$. As argued in the previous paragraph, we will gain a factor of $q^{(y-1)+k-3+i_1+2} = q^{(y-1)+k-1+i_i}$.

We continue the above placement scheme, skipping over positions where descents are already in σ. The last position will contribute $q^{(y-1)+n-1}$, and the sum over all positions for n in σ which increase des yields $q^{y+k-2} + q^{y+k-1} + \cdots + q^{y+n-2} = q^{y+k-2} \times \{1 + q + q^2 + \cdots + q^{n-k}\} = q^{y+k-2}[n-k+1]$. Now summing over all $\sigma \in S_{n-1}$ with $k-2$ descents yields the second term in the recurrence.

□

We have the following easy lemma.

Lemma 5.2. *We have*

$$E_{n,k}(y,q) = [y+k-1]E_{n-1,k}(y,q) + q^{y+k-2}[n-k+1]E_{n-1,k-1}(y,q)$$

for $n, k \in \mathbb{N}$.

Proof. Let $B = \mathbb{T}_n$ in Lemma 2.2. □

We can now prove the following theorem.

Theorem 5.3. *For any* $n, k \in \mathbb{N}$ *we have that* $\tilde{E}_{n,k}(y,q) = E_{n,k}(y,q)$.

Proof. It is clear that $\tilde{E}_{1,1}(y,q) = [y]$, and it is easy to check by definition of $A_{1,1}(y,q,\mathbb{T}_1)$ that $E_{1,1}(y,q) = [y]$. Thus the $\tilde{E}_{n,k}(y,q)$ and the $E_{n,k}(y,q)$ satisfy the same initial conditions, and they satisfy the same recurrence by Proposition 5.1 and Lemma 5.2. Hence $\tilde{E}_{n,k}(y,q) = E_{n,k}(y,q)$ for all $n, k \in \mathbb{N}$. □

An immediate corollary of Theorem 5.3 is the following.

Proposition 5.4. *The pair* $(\ell rmin(-), (n - \ell rmin(-))(y-1) + maj(-))$ *is cycle-Mahonian.*

Proof. By definition,

$$\sum_{\sigma \in S_n} [y]^{\ell rmin(\sigma)} q^{(n-\ell rmin(\sigma))(y-1)+maj(\sigma)} = \sum_{k=1}^{n} \tilde{E}_{n,k}(y,q).$$

By Theorem 5.3,

$$\sum_{k=1}^{n} \tilde{E}_{n,k}(y,q) = \sum_{k=1}^{n} E_{n,k}(y,q).$$

Again by definition,

$$\sum_{k=1}^{n} E_{n,k}(y,q) = \sum_{k=1}^{n} A_{n,k-1}(y,q,\mathbb{T}_n),$$

which is equal to $[y][y+1]\cdots[y+n-1]$ by (4) (since $A_{n,n}(y,q,\mathbb{T}_n)=0$). □

Note that if we consider the triangular board $\mathbb{T}_n \subset SQ_n$, we can bijectively associate to a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ with k descents a placement of n rooks on SQ_n such that exactly k rooks lie off $\mathbb{T}_n$ in the following way (first noted in [10]). First, we find the product $y_1y_2\cdots y_p$ of cycles as was done when computing $\ell rmin(-)$ earlier in this section. Then we place a rook on square (i,j) of SQ_n if and only if i follows j in one of the cycles y_ℓ. It is easy to verify that this placement will have exactly k rooks off $\mathbb{T}_n$, and that this procedure can be reversed. This placement is the *descent graph* of σ, which we will denote $DG(\sigma)$. Note that by Theorem 4.1 and the above discussion, we now have that

$$E_{n,k}(y,q) = \sum_{\sigma \in S_n,\ des(\sigma)=k-1} [y]^{cyc(DG(\sigma))} q^{(n-cyc(DG(\sigma))(y-1)+s_{\mathbb{T}_n,b}(DG(\sigma))+E(DG(\sigma))}.$$

We can now prove the following.

Theorem 5.5. *For any $n,k \in \mathbb{N}$*

$$\sum_{\sigma \in S_n,\ des(\sigma)=k,\ cyc(DG(\sigma))=\ell} q^{s_{\mathbb{T}_n,b}(DG(\sigma))+E(DG(\sigma))} = \sum_{\sigma \in S_n,\ des(\sigma)=k,\ \ell rmin(\sigma)=\ell} q^{maj(\sigma)}.$$

Proof. We know that

$$\sum_{\sigma \in S_n,\ des(\sigma)=k} [y]^{cyc(DG(\sigma))} q^{(n-cyc(DG(\sigma))(y-1)+s_{\mathbb{T}_n,b}(DG(\sigma))+E(DG(\sigma))} = E_{n,k+1}(y,q). \tag{5}$$

By Theorem 5.3, (5) is equal to $\tilde{E}_{n,k+1}(y,q)$, where

$$\tilde{E}_{n,k+1}(y,q) = \sum_{\sigma \in S_n,\ des(\sigma)=k} [y]^{\ell rmin(\sigma)} q^{(n-\ell rmin(\sigma))(y-1)+maj(\sigma)},$$

and hence

$$\sum_{\sigma \in S_n,\ des(\sigma)=k} [y]^{cyc(DG(\sigma))} q^{(n-cyc(DG(\sigma))(y-1)+s_{\mathbb{T}_n,b}(DG(\sigma))+E(DG(\sigma))} =$$

$$\sum_{\sigma \in S_n,\ des(\sigma)=k} [y]^{\ell rmin(\sigma)} q^{(n-\ell rmin(\sigma))(y-1)+maj(\sigma)}. \tag{6}$$

If we let $z = [y]q^{-(y-1)}$ in (6), then we have that

$$q^{n(y-1)} \sum_{\sigma \in S_n,\ des(\sigma)=k} z^{cyc(DG(\sigma))} q^{s_{\mathbb{T}_n,b}(DG(\sigma))+E(DG(\sigma))} = q^{n(y-1)} \sum_{\sigma \in S_n,\ des(\sigma)=k} z^{\ell rmin(\sigma)} q^{maj(\sigma)}.$$

Thus

$$\sum_{\sigma \in S_n,\ des(\sigma)=k} z^{cyc(DG(\sigma))} q^{s_{\mathbb{T}_n,b}(DG(\sigma))+E(DG(\sigma))}$$

and

$$\sum_{\sigma \in S_n,\ des(\sigma)=k} z^{\ell rmin(\sigma)} q^{maj(\sigma)}$$

are equal polynomials in the variable z over $\mathbb{N}[q]$, and hence equal powers of z must have equal coefficients. In particular the coefficient of z^ℓ in each must be equal. That is

$$\sum_{\sigma \in S_n,\ des(\sigma)=k\ cyc(DG(\sigma))=\ell} q^{s_{\mathbb{T}_n,b}(DG(\sigma))+E(DG(\sigma))} = \sum_{\sigma \in S_n,\ des(\sigma)=k,\ \ell rmin(\sigma)=\ell} q^{maj(\sigma)}$$

as desired.

□

Recall that a permutation statistic s on S_n is called *Euler-Mahonian* if the pairs (des, s) and (des, maj) have the same distribution on S_n, that is,

$$\sum_{\sigma \in S_n,\ des(\sigma)=k} q^{s(\sigma)} = \sum_{\sigma \in S_n,\ des(\sigma)=k} q^{maj(\sigma)}$$

for all values of k. Theorem 5.5 leads us to define the following generalization. We will say a pair of permutation statistics $(s_1(-), s_2(-, y))$ is *cycle-Euler-Mahonian* if it is cycle-Mahonian as defined in Section 4, and

$$\sum_{\sigma \in S_n,\ des(\sigma)=k,\ s_1(\sigma)=\ell} q^{s_2(\sigma,1)} = \sum_{\sigma \in S_n,\ des(\sigma)=k,\ \ell rmin(\sigma)=\ell} q^{maj(\sigma)}. \tag{7}$$

This definition generalizes that of Euler-Mahonian, because if $(s_1(-), s_2(-, y))$ satisfies (7) then

$$\sum_{\sigma \in S_n,\ des(\sigma)=k} q^{s_2(\sigma,1)} = \sum_{\ell} \left\{ \sum_{\sigma \in S_n,\ des(\sigma)=k,\ s_1(\sigma)=\ell} q^{s_2(\sigma,1)} \right\} =$$
$$\sum_{\ell} \left\{ \sum_{\sigma \in S_n,\ des(\sigma)=k,\ \ell rmin(\sigma)=\ell} q^{maj(\sigma)} \right\} = \sum_{\sigma \in S_n,\ des(\sigma)=k} q^{maj(\sigma)}.$$

Thus if $(s_1(-), s_2(-, y))$ is cycle-Euler-Mahonian, this implies that $s_2(-, 1)$ is Euler-Mahonian.

By Corollary 4.2 $(cyc(DG(-)), (n-cyc(DG(-)))(y-1)+s_{\mathbb{T}_n,b}(DG(-))+E(DG(-)))$ is cycle-Mahonian, and by Theorem 5.5 we see that

$$(des(-), cyc(DG(-)), s_{\mathbb{T}_n,b}(DG(-)) + E(DG(-)))$$

and

$$(des(-), \ell rmin(-), maj(-))$$

have the same distribution. Thus $(cyc(DG(-)), (n-cyc(DG(-)))(y-1)+s_{\mathbb{T}_n,b}(DG(-))+E(DG(-)))$ is an example of a cycle-Euler-Mahonian pair of statistics on S_n.

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