Cycle-counting p, q-rook theory and the p, q, y-Stirling numbers of the second kind

Fred Butler
West Virginia University
fbutler@math.wvu.edu
www.math.wvu.edu/~fbutler
Joint Meeting of the AMS, DMV, and ÖMG

16 June 2005

Terminology and Notation

- We let SQ_n denote the $n \times n$ chess board, with squares numbered as below.

- A *board* is any subset of squares of SQ_n for some $n \in \mathbb{N}$ (represented in our pictures by unshaded squares).
- A *Ferrers board* is a board with columns $b_1 \leq \cdots \leq b_n$ of non-decreasing height from left to right, denoted $B(b_1, \ldots, b_n)$ (in pictures of a Ferrers board $B \subseteq SQ_n$, we omit the squares of SQ_n which are not part of B).

Classical Rook Numbers (Kaplansky and Riordan)

- Recall that in chess, a rook attacks all squares in its row and column.

- We call an arrangement of k **non-attacking** rooks on the squares of a board $B \subseteq SQ_n$ a *rook placement*; the number of such arrangements, called the *kth rook number* of B, is denoted $r_k(B)$.

- Equivalently, $r_k(B)$ is the number of different ways to arrange k rooks on B such that no two rooks are in the same row or column of SQ_n.

- **Classical Factorization Theorem (Goldman, Joichi, and White 1975)** If $B = B(b_1, \ldots, b_n) \subseteq SQ_n$, then

$$\sum_{k=0}^{n} r_{n-k}(B) x(x-1)\cdots(x-k+1) = \prod_{i=1}^{n}(x+b_i-i+1).$$

- Much work has been done by replacing $r_k(B)$ by some weighted count of k rook placements

$$\sum_{\substack{P,\ k \text{ rook} \\ \text{placements on } B}} weight(P),$$

in such a way that versions of the **Factorization Theorem** and other known theorems hold.

q-Analogs

- A q-analog of a combinatorial quantity is an expression involving the parameter q such that when $q = 1$ (or $q \to 1$), we get the original quantity.
- For $n \in \mathbb{N}$, $[n]_q := 1+q+\cdots+q^{n-1}$; note that $[n]_1 = n$.
- Generalizing the above, for $x \in \mathbb{R}$

$$[x]_q := \frac{1-q^x}{1-q};$$

  note that $\lim_{q\to 1}[x]_q = x$.
- We define $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$.
- Finally for $z \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$\begin{bmatrix} z \\ k \end{bmatrix}_q := \frac{[z]_q[z-1]_q \cdots [z-k+1]_q}{[k]_q!}.$$

p,q-Analogs

- A p,q-analog is an expression in the parameters p and q such that letting $p = 1$ (or $p \to 1$) gives the q-analog.
- For $x \in \mathbb{R}$

$$[x]_{p,q} := \frac{p^x - q^x}{p - q};$$

note when $n \in \mathbb{N}$, $[n]_{p,q} = (p^n - q^n)/(p - q) = p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1}$.

- Note that $[x]_{1,q} = [x]_q$.
- We can analogously define

$$[n]_{p,q}! := [n]_{p,q}[n-1]_{p,q} \cdots [1]_{p,q}$$

and

$$\begin{bmatrix} z \\ k \end{bmatrix}_{p,q} := \frac{[z]_{p,q}[z-1]_{p,q} \cdots [z-k+1]_{p,q}}{[k]_{p,q}!}$$

Cycle-counting p, q-rook numbers

- We can now define the *kth cycle-counting p, q-rook number* of a **Ferrers board** B, denoted $R_k(y, p, q, B)$, by the equation

$$R_k(y,p,q,B) = \sum_{\substack{P,\ k \text{ rook} \\ \text{placements on } B}} [y]_{p,q}^{cyc(P)} q^{\alpha_B(P)+\epsilon_B(P)+E(P)(y-1)} \times p^{\beta_B(P)-(c_1(P)+\cdots+c_k(P))+A(P)(y-1)}.$$

- This was build upon rook theory models of Garsia and Remmel (q-rook numbers), Chung and Graham (cycle-counting rook numbers), Ehrenborg, Haglund, Readdy (cycle-counting q-rook numbers), and Remmel and Wachs (p, q-rook numbers).

- For a placement P of k non-attacking rooks on a Ferrers board B, we let each rook from P cancel all squares to the right in its row. Then

  1. $\alpha_B(P)$ = number of uncancelled squares of B above a rook,
  2. $\beta_B(P)$ = number of uncancelled squares of B below a rook,
  3. $\epsilon_B(P)$ = number of uncancelled squares of B in an empty column.

- For $B \subseteq SQ_n$, we can associate to P a directed graph G_P on n vertices labeled $1, 2, \ldots n$, where there is a directed edge in G_P from i to j iff there is a rook from P on square (i, j).
- Then $cyc(P)$ is defined to equal the number of cycles in the directed graph G_P.

- $E(P)$ and $A(P)$ are technical statistics having to do with squares which create new cycles in the directed graph G_P.
- $c_i(P)$ is the column label of the ith rook from the left from P.

- **Cycle-Counting** p, q**-Factorization Theorem (2004)**
If $B = B(b_1, \ldots, b_n) \subseteq SQ_n$, then

$$\sum_{k=0}^{n} R_{n-k}(y,p,q,B)p^{(n-k)x+\binom{n-k+1}{2}}[x]_{p,q}[x-1]_{p,q}\ldots[x-k+1]_{p,q}$$

$$\prod_{b_i<i}[x+b_i-i+1]_{p,q}\prod_{b_i\geq i}[x+b_i-i+y]_{p,q}.$$

- If we let $y, p, q \to 1$ in the above equation, we get the **Classical Factorization Theorem** of Goldman, Joichi, and White:

$$\sum_{k=0}^{n} r_{n-k}(B)x(x-1)\cdots(x-k+1) = \prod_{i=1}^{n}(x+b_i-i+1).$$

- The $R_k(y,p,q,B)$ also satisfy the following versions of known identities for classical rook numbers.
  1. If $B = B(b_1,\ldots,b_n)$ is a Ferrers board, $B' = B(b_1,\ldots b_n, b_{n+1})$ and $b_{n+1} \geq n+1$, then
  $$R_k(y,p,q,B') = q^{b_{n+1}-k+y-1} R_k(y,p,q,B) + p^{-(n+1)}[b_{n+1}-k+y]_{p,q} R_{k-1}(y,p,q,B).$$
  2. For $B = B(b_1,\ldots,b_n)$ a Ferrers board with $b_i \geq i$,
  $$R_{n-k}(y,p,q,B) p^{\binom{n-k+1}{2}+k(n-k)} = \frac{1}{[k]_{p,q}!} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_{p,q} (-1)^{k-j} q^{\binom{k-j}{2}} p^{(n-k)(k-j)+\binom{k-j+1}{2}} \times \prod_{i=1}^{n} [j+b_i-i+y]_{p,q}.$$

p, q, y-Stirling numbers of the second kind

- It is known that for the "triangular" Ferrers board $B(0, 1, \ldots, n-1) \subseteq SQ_n$,

  $r_{n-k}(B(0, 1, \ldots, n-1)) = S_{n,k} =$ number of partitions of

  $$\{1, 2, \ldots, n\} \text{ into } k \text{ blocks,}$$

  the Stirling number of the second kind.

- There is a well-known bijection, which sends a placement P of $n-k$ rooks on $B(0, 1, \ldots, n-1)$ to a partition S of $\{1, 2, \ldots, n\}$ into k parts, where i and j are in the same block of S iff there is a rook from P on square (i, j)

- This fact is often used to define or study generalizations of the $S_{n,k}$.
- We can define a p, q, y-version of the $S_{n,k}$ in a similar manner, but we need to be a little careful.
- Note that for the board $B(0, 1, \ldots n-1)$, we have that $b_i < i$ for each i; it is helpful for us to have a Ferrers board such that $b_i \geq i$.
- For this reason, we work instead with the board $B(1, 2, \ldots n-1) \subseteq SQ_{n-1}$.
- The various rook numbers of the board $B(1, 2, \ldots n-1)$ are closely related to those of $B(0, 1, \ldots n-1)$ (in fact the classical rook numbers of these two boards are exactly the same).

- We can then define the *kth p, q, y-Stirling number of the second kind* by the equation

$$S_{n,k}(y,p,q) := p^{-(n-k)} R_{n-k}(y,p,q,B(1,2,\ldots,n-1)).$$

- We see that these numbers satisfy several versions of known identities for the classical Stirling numbers.

1. $$S_{n+1,k}(y,p,q) = q^{y+k-2} S_{n,k-1}(y,p,q) + p^{-(n+1)}[y+k-1]_{p,q} S_{n,k}(y,p,q)$$
$$\Longleftrightarrow$$
$$S_{n+1,k} = S_{n,k-1} + k S_{n,k}.$$

$$2. \quad \sum_{k=0}^{n} S_{n,k}(y,p,q) p^{(n-k)x+\binom{n-k+1}{2}} \times$$

$$[x]_{p,q}[x-1]_{p,q} \cdots [x-k+1]_{p,q} = [x]_{p,q}[x+y-1]_{p,q}^{n-1}$$

$$\Longleftrightarrow$$

$$\sum_{k=0}^{n} S_{n,k} x(x-1) \cdots (x-k+1) = x^n.$$

$$3.\quad S_{n,k}(y,p,q)p^{\binom{n-k+1}{2}+k(n-k)} =$$

$$\frac{1}{[k]_{p,q}!}\sum_{j=1}^{k}\begin{bmatrix}k\\j\end{bmatrix}_{p,q}(-1)^{k-j}q^{\binom{k-j}{2}}p^{(n-k)(k-j)+\binom{k+j+1}{2}}\times$$

$$[y+j-1]_{p,q}^{n-1}[j]_{p,q}$$

$$\Longleftrightarrow$$

$$S_{n,k} = \frac{1}{k!}\sum_{j=1}^{k}\binom{k}{j}(-1)^{k-j}j^{n}$$

Statistics

- We can also look at the various statistics in terms of set partitions; in order to compute the statistics on a partition S of $\{1, 2, \ldots n\}$ into k blocks, do the following:

  1. Associate S to a placement P' of $n-k$ rooks on $B(0, 1, \ldots, n-1) \subseteq SQ_n$ via the earlier bijection.
  2. Associate P' to the obvious $n-k$ rook placement P on the board $B(1, 2, \ldots, n-1) \subseteq SQ_{n-1}$.
  3. For any of the statistics discussed in this paper, we can define $stat(S) := stat(P)$.

- The statistic $cyc(S)$ counts the number of pairs $(i, i+1)$, $1 \le i \le n-1$, where i and $i+1$ are in the same part of S (corresponding to the number of columns of $B(1, 2, \ldots, n-1)$ with a rook on the diagonal).
- The statistic $E(S)$ counts the number of pairs $(i, i+1)$, $1 \le i \le n-1$, where i and $i+1$ are *not* in the same part of S (corresponding to the number of columns of $B(1, 2, \ldots, n-1)$ *without* a rook on the diagonal)..
- Finally,

$$c_1(S) + \cdots + c_{n-k}(S) = k - n + \sum_{\substack{1 \le i \le n, i \text{ not smallest} \\ \text{number in its part of } S}} i.$$

References

[1] K. Briggs and J. Remmel. A p, q-analogue of a formula of Frobenius. *Electron. J. Combin.*, 10:#R9, 2003.

[2] F. Butler. *Cycle Counting q-Rook Theory and Other Generalizations of Classical Rook Theory.* PhD thesis, University of Pennsylvania, 2004.

[3] F. Butler. Rook theory and cycle-counting permutations statistics. *Adv. in Appl. Math.*, 33:655–675, 2004.

[4] F. Chung and R. Graham. On the cover polynomial of a digraph. *J. Combin. Theory, Ser. B*, 65:273–290, 1995.

[5] M. Dworkin. Factorization of the cover polynomial. *J. Combin. Theory, Ser. B*, 71:17–53, 1997.

[6] M. Dworkin. An interpretation for Garsia and Remmel's q-hit numbers. *J. Combin. Theory, Ser. A*, 81:149–175, 1998.

[7] R. Ehrenborg, J. Haglund, and M. Readdy. Colored juggling patterns and weighted rook placements. Unpublished manuscript.

[8] A. Garsia and J. Remmel. q-Counting rook configurations and a formula of Frobenius. *J. Combin. Theory, Ser. A*, 41:246–275, 1986.

[9] J. Goldman, J. Joichi, and D. White. Rook theory I: Rook equivalence of Ferrers boards. *Proc. Amer. Math. Soc.*, 52:485–492, 1975.

[10] J. Haglund. Rook theory and hypergeometric series. *Adv. in Appl. Math.*, 17:408–459, 1996.

[11] J. Haglund. q-Rook polynomials and matrices over finite fields. *Adv. in Appl. Math.*, 20:450–487, 1998.

[12] I. Kaplansky and J. Riordan. The problem of the rooks and its applications. *Duke Math. J.*, 13:259–268, 1946.

[13] J. Remmel and M. Wachs. Generalized p, q-Stirling numbers. Unpublished manuscript.

[14] R. Stanley. *Enumerative Combinatorics Volume I.* Cambridge University Press, 1986.